What is non-commutative harmonic analysis?

The following essay is a review of the book *Real reductive groups I*, by Nolan R. Wallach. This review offers an expository account of the subject of harmonic analysis and representation theory of real reductive Lie groups. Wallach's book lays the groundwork for an eventual *Part* II which will culminate with Harish-Chandra's proof of the explicit Plancherel formula on any real reductive Lie group. As a member of the new generation of representation theorists, I am somewhat embarrassed to admit my ignorance of many details in Harish-Chandra's pioneering program; the reader is directed to the essays of Howe, Varadarajan and Wallach in [2] for accounts of Harish-Chandra's work. Instead, I will explain to the interested reader just what "harmonic analysis and representation theory of real reductive groups" means in the context of Wallach's book.

What is a real reductive group?

We begin with an exercise to convince the reader that he is already well acquainted with the notion of a *real reductive group*, at least in spirit, if not in name. The exercise is this: Compile a list of groups which are simultaneously smooth manifolds (so-called *Lie groups*). The real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, $\mathbb{R}^n$ and $\mathbb{C}^n$ fit into this list; as well as the *n-torus* $(S^1 \times \cdots \times S^1)$. The *matrix groups* provide an abundant source of noncommutative examples. The groups, $Gl_n\mathbb{R}$ ($n \times n$ real invertible matrices), $Sl_n\mathbb{R}$ (the determinant $= 1$ subgroup of $Gl_n\mathbb{R}$), or the invariance group of a nondegenerate form are noncommutative Lie groups which arise in any undergraduate linear algebra course. More generally, any topologically closed subgroup G of $Gl_n\mathbb{F}$ ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$) which is closed under the operation of conjugate transpose is of the desired type and we take this as the definition of a *real reductive group* G. These are the basic objects of study in Wallach's book.

One might ask if every matrix group is a real reductive group. The answer is no; for instance, $\{g \in Sl_n\mathbb{R}\colon g \text{ is upper triangular}\}$ is not a real reductive group, rather a *solvable* group. The *Levi Mal'cev Theorem* insures that every smooth group of matrices can be written as a semidirect product of a solvable group and a real reductive group.

Where the subject of the book comes from

We begin with Fourier analysis on the unit circle S^1. Recall the elementary exponentials $\gamma_n(\theta) = \exp(\text{in } \theta)$ on S^1, where $i = (-1)^{1/2}$ and $n \in \mathbb{Z}$. Given a smooth function f on S^1, we define the *Fourier transform* $\Theta_{\gamma_n}(f) = \int_{S^1} f(\theta)\gamma_n(\theta)d\theta$. Then Θ_{γ_n} can be viewed as a continuous linear function (a *distribution*) on $C^\infty(S^1)$. Following the viewpoint of Harish-Chandra [1], we will view the process of writing the *Dirac distribution* $\delta(\delta(f) = f(e))$ as a "linear combination of Θ_{γ_n}, $n \in \mathbb{Z}$" as the *explicit Plancherel problem for* S^1. In our setting, the solution to this problem is $\delta = \sum_{n\in\mathbb{Z}} \Theta_{\gamma_n}$, which amounts to the equality of f with its Fourier series at the identity.

Suppose we replace S^1 by a real reductive group G and try to carry through the above discussion. We begin by finding an analogue of the Fourier transform. Notice that $\mathcal{E} = \{\gamma_n : n \in \mathbb{Z}\}$ exhausts the set of continuous group homomorphisms of S^1 into S^1. For our general G, one might replace $\mathcal{E}$ by the set of continuous homomorphisms of G into S^1, denoted $\widehat{G}_p$ and referred to as the *Pontryagin dual of* G. This approach succeeds when G is commutative; in fact, this will work for any locally compact commutative topological group. This success is due, in part, to the fact that there are enough characters to *separate points*. However, once we move into the noncommutative setting it may well happen that $\widehat{G}_p = \{e\}$.

Altering our viewpoint, we refer to a continuous group homomorphism π from G into $\mathcal{U}(H_\pi)$ = unitary operators on the Hilbert space H_π as a *unitary representation of* G. In particular, $\pi(g)$ is a unitary operator on H_π for every $g \in G$. We call H_π the *representation space of* π and a subspace $S \subseteq H_\pi$ is called π-*invariant* if $\pi(g)S \subseteq S$, for all $g \in G$. Those π for which H_π admits no closed π-invariant subspaces other than 0 and H_π are called *irreducible unitary representations* and we use the notation $\widehat{G}_u$ for the set of all such representations. As evidence that we are proceeding along the correct lines, the *Gelfand-Raikov Theorem* tells us that there are sufficiently many irreducible unitary representations to separate points. Additionally, in the commutative case, any $\chi \in \widehat{G}_p$ can be viewed as a one dimensional irreducible unitary representation $\chi : G \to \mathcal{U}(\mathbb{C})$, acting by multiplication and this yields a natural bijection $\widehat{G}_p = \widehat{G}_u$.

What is the price we have paid by replacing $\widehat{G}_p$ by $\widehat{G}_u$? This can be answered in many ways, but the bottom line is that the Hilbert spaces H_π

associated to $\pi \in \widehat{G}_u$ may be infinite dimensional. Nevertheless, ignoring this point, we can still mimic the prior definitions and set $\pi(f) = \int_G f(g)\pi(g)dg$, for smooth compactly supported functions f, with $\pi \in \widehat{G}_u$ and dg a fixed Haar measure on G. With some care and work, we find that $\pi(f)$ defines a trace class bounded operator on H_π and we obtain a complex number by setting the *Fourier transform* of f at π equal to $\Theta_\pi(f) = \text{trace} \int_G f(g)\pi(g)dg$. In particular, Θ_π can be viewed as a distribution on G. Following [1], we can now state

(1) **Explicit Plancherel problem for** G. *Write the Dirac distribution* ($\delta(f) = f(e)$) *as an explicit "linear combination of* Θ_π*'s, with* $\pi \in \widehat{G}_u$*."*

Primary goal of the book

We take the solution of the explicit Plancherel problem as our guiding light. However, the above discussion suggests that a foray into the representation theory of a real reductive group will be necessary. For starters, it would seem we will need to have in hand a parametrization of $\widehat{G}_u$ if there is any hope of explicitly solving (1). To see what is at stake here, an *abstract* (as opposed to *explicit*) Plancherel Formula asserts that $f(e) = \int_{\widehat{G}_u} \Theta_\pi(f)d\mu(\pi)$, where $d\mu$ is the *Plancherel measure* on the set $\widehat{G}_u$. For example, if $G = S^1$, then $d\mu$ is just counting measure on $\mathbb{Z} = \widehat{G}_u$, and this integral becomes a Fourier series. The desire for explicitness in (1) amounts to a good description of the Plancherel measure in terms of data intrinsically attached to G.

In large part, *Real reductive groups*.I lays out a first approximation to the parametrization of $\widehat{G}_u$. This classification appeared in the 1970s and parametrizes the somewhat larger class of *irreducible admissible representations*, denoted $\widehat{G}_a$. In the definition of an *admissible representation* (π, H_π) we require only that $\pi(g)$ be a linear automorphism of a Hilbert space H_π, then take as a defining axiom a key property exhibited by any $\pi \in \widehat{G}_u : \pi$ restricted to a *maximal compact subgroup* K of G decomposes into a Hilbert space direct sum of irreducible representations of K and each class of irreducible K-representation occurs at most finitely often in this sum. The *Langlands classification* will parametrize $\widehat{G}_a$ in terms of the irreducible tempered representations (defined below) of reductive subgroups L of G, while the *Knapp-Zuckerman classification* then provides a list of the irreducible tempered representations of each such L. In the end, the support of the Plancherel measure lies inside the set $\widehat{G}_t$ of irreducible tempered representations of G, which in turn lies inside $\widehat{G}_u$. See Figure 1.

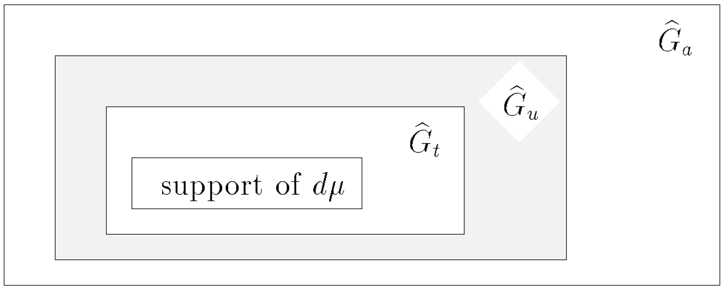


FIGURE 1

BUILDING REPRESENTATIONS

We need two fundamental representation theoretic constructions for building new representations from old. First, a process of going from admissible representations of reductive subgroups L of G to admissible representations of G, a technique referred to as *induction.* Secondly, a "reverse" process of going from admissible representations of G to admissible representations of reductive subgroups L of G, a procedure we shall term *reduction.* Each technique is used in the proof of the classification.

Having fixed our ambient real reductive group G, we need to describe a canonical family of reductive subgroups L of G. The idea is best illustrated by looking at the case when $G = Sl_n\mathbb{R}$. In this case, we define P_m to be the subgroup of upper triangular matrices. Any subgroup P of G (including G itself) containing P_m will be called a *parabolic subgroup.* In our special case, a typical P will consist of "block" upper triangular matrices; for example, if $n = 3$, we can schematically describe four parabolic subgroups as

$$\begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix} \quad \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \quad \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \quad \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}$$

In general, parabolic subgroups are not reductive, but we can always decompose P as a semidirect product $P = LN$, where N is a normal nilpotent subgroup of P and L is a reductive subgroup of G. The subgroups L arising from parabolic subgroups of G will be crucial in all that follows and we refer to them as *Levi subgroups*.

Given an admissible representation (τ, H_τ) of a Levi subgroup L, we extend τ to a representation of $P = LN$, by letting $\tau(n)$ be the identity operator for every $n \in N$. Forming the function space

$$\begin{aligned}\mathrm{ind}_P(\tau) \underset{\mathrm{def}}{=} \{f : G \to H_\tau | f \text{ is continuous and } f(gp) \\ = \tau(p)^{-1} f(g), g \in G, p \in P\},\end{aligned}$$

we then define π_{reg} on $\mathrm{ind}_P(\tau)$ via $[\pi_{\mathrm{reg}}(x)f](g) = f(x^{-1}g)$, x, $g \in G$. By completing $\mathrm{ind}_P(\tau)$ with respect to a particular norm we arrive at $\mathrm{Ind}_P(\tau)$, referred to as a *P-induced representation of G*. All such induced representations are admissible representations of G and this procedure exhibits the process of induction central to the Langlands classification. In fact, one can view this induction construction as an old friend from *vector bundle theory*: If (τ, H_τ) is a finite dimensional representation of a parabolic subgroup P, then one can associate a homogeneous vector bundle $\mathcal{V}_\tau$ over the manifold G/P, having fibre H_τ at the identity coset. The space $\mathrm{ind}_P(\tau)$ describes the continuous sections of this vector bundle.

Before describing our reduction procedure, we need a fundamental idea of Harish-Chandra. Namely, that we replace an admissible representation π on the Hilbert space H_τ by the subspace $H_{\pi,K} \subseteq H_\pi$ consisting of the algebraic (as opposed to Hilbert space) direct sum of irreducible representations of K occurring in the restriction of π to a maximal compact subgroup K of G. The resulting vector space $H_{\pi,K}$, stripped of its topology, is called the underlying *Harish-Chandra module* of π; the study of such modules is initiated in Chapter 3 of the book. Of course, this process of passing from π to $H_{\pi,K}$ comes at a price: Harish-Chandra modules are not representation spaces for the group G. However, the price is not too high: The subspace $H_{\pi,K}$ is dense in H_π, simultaneously carrying a representation of the complexified Lie algebra $\mathfrak{g}$ of G and our maximal compact group K. Moreover, the notions of irreducibility of π and irreducibility of $H_{\pi,K}$ are equivalent and nothing will be lost if we proceed to classify the irreducible Harish-Chandra modules.

To describe our reduction technique, let $\mathfrak{n}$ denote the complexified Lie algebra of the Lie group N. We consider the functor which assigns to an admissible representation π of G the space $H_0(\mathfrak{n}, \pi) \underset{\mathrm{def}}{=} H_{\pi,K}/(\mathfrak{n}.H_{\pi,K})$, referred to as the space of $\mathfrak{n}$-*coinvariants* in π. This defines a right exact covariant functor from the category $\mathcal{HC}_G$ of Harish-Chandra modules for G into the category of complex vector spaces, so we may study the higher order derived functors applied to π, which are referred to as the $\mathfrak{n}$-*homology groups* with *coefficients in* π and denoted $H_k(\mathfrak{n}, \pi)$, $k \geq 0$. Each of the vector spaces $H_k(\mathfrak{n}, \pi)$ will be the Harish-Chandra module of an admissible (not necessarily irreducible) representation of L; this is the reduction technique needed in the sequel.

The subrepresentation theorem

One of the first results giving us some control over the set of irreducible admissible representations of G is Harish-Chandra's *Subquotient Theorem* (Chapter 3). This result asserts that to each irreducible admissible representation π of G, we may associate a finite-dimensional representation τ_π of the Levi subgroup L_m so that π occurs as a *subquotient* of $\mathrm{Ind}_{Pm}(\tau_\pi)$; this means there exist closed π-invariant subspaces $S \subseteq S' \subseteq \mathrm{Ind}_{Pm}(\tau_\pi)$ so that $S'/S = H_\pi$. The numerous consequences of this result include (in Wallach's treatment) Casselman's subrepresentation theorem:

(2) **Theorem.** *Each irreducible admissible representation* π *is a closed* π*-invariant subspace of some* $\mathrm{Ind}_{P_m}(\tau_\pi)$.

In fact, every occurrence of an irreducible π as a subrepresentation of a P-induced representation $\mathrm{Ind}_P(\tau)$ is determined by the Harish-Chandra module $H_0(\mathfrak{n}, \pi)$ for the Levi subgroup L. To see this, one introduces a suitable notion of morphism in categories of Harish-Chandra modules and proves for an admissible representation π of G:

(3) **Frobenius reciprocity.** $\mathrm{Hom}_G(H_{\pi,K}, \mathrm{Ind}_P(\tau)_K)$
$= \mathrm{Hom}_L(H_0(\mathfrak{n}, \pi), H_{\tau,K\cap L})$.

This provides the fundamental connection between our induction and reduction techniques. Also, it should be noted that if π is irreducible, then any nonzero element of the left-hand side of (3) is necessarily injective.

The proof of Frobenius reciprocity is surprisingly elementary and given this $\mathfrak{n}$-homological reformulation, Casselman's subrepresentation theorem together with the other embedding statements follows from

(4) **Theorem.** *If* $\pi \in \widehat{G}_a$, *then* $H_0(\mathfrak{n}_m, \pi) \neq 0$.

In addition, as will soon become apparent, the proof of Langland's classification depends on a careful study of $H_0(\mathfrak{n}_m, \pi)$. Ultimately, we can approach (4) from three different directions: algebraically, via matrix coefficients or using the Fourier transform Θ_π. By the end of the review, we will have touched on each of these viewpoints, but for now we will begin with the algebraic perspective. Recalling the Levi subgroup L_m associated to P_m, we can further decompose $L_m = M_m A_m$, with M_m compact and A_m diffeomorphic to a finite product of copies of the multiplicative group of positive real numbers $\mathbb{R}^\times$; we can analogously decompose $P = MAN$ (but M won't necessarily be compact). By restriction, viewing $H_0(\mathfrak{n}_m, \pi)$ as a finite-dimensional representation Ω of the commutative group A_m, there must exist Ω-invariant subspaces

$$0 = S_0 \subseteq S_1 \subseteq S_2 \subseteq \cdots \subseteq S_t = H_0(\mathfrak{n}_m, \pi)$$

such that each subquotient S_j/S_{j-1} is a one-dimensional character $e^{\nu(\log a)}$, $a \in A_m$, ν in the dual space a_m^* of the complexified Lie algebra of A_m. This allows us to define

$$\mathcal{A}_{\text{hom}}(\pi) = \{\nu : e^{\nu(\log a)} \text{ arises as a subquotient } S_j/S_{j-1} \text{ of } \Omega\},$$

referred to as *homology exponents of* π.

The *a priori* algebraic problem of showing that $\mathcal{A}_{\text{hom}}(\pi) \neq 0$ is amazingly deep. One argument, the original approach of Casselman, relies on an analysis of *matrix coefficients* (the topic of Chapter 4) associated to the Harish-Chandra module of π. A matrix coefficient $f_{v,\omega}$ is a real analytic function on G built from a vector v in $H_{\pi,K}$ and a K-finite vector ω in the algebraic dual space of $H_{\pi,K}$, via the rule $f_{v,\omega}(g) = \langle \omega, \pi(g)v \rangle$, $g \in G$. Harish-Chandra's *theory of the constant term* leads one to analyze the growth behavior of matrix coefficients on G. Recalling that $A_m = (\mathbb{R}^\times \times \cdots \times \mathbb{R}^\times)$, we define A_m^- as the image under the exponential map of a certain open convex cone $(\mathfrak{a}_{m,0}^-) \subseteq \mathfrak{a}_{m,0}$; the situation for $Sl_3\mathbb{R}$ is indicated in Figure 2. One is ultimately reduced to the growth analysis of matrix coefficients on closure (A_m^-). Matrix coefficients satisfy differential equations, arising from the action of certain left invariant differential operators on G on the matrix coefficients. Solving these equations, using something not unlike the classical *Frobenius method* for ordinary differential equations with a regular singular point, we find that $f_{v,\omega}$ has an expansion in exponentials of the form $e^{\nu(\log a)}$, for $a \in A_m^-$, $\nu \in (\mathfrak{a}_m)^*$. This allows us to define the *asymptotic exponents of* π:

$$\mathcal{A}_{\text{asy}}(\pi) = \{\nu : e^{\nu(\log a)} \text{ arises in the expansion of some } f_{v,\omega} \text{ for } \pi\}.$$

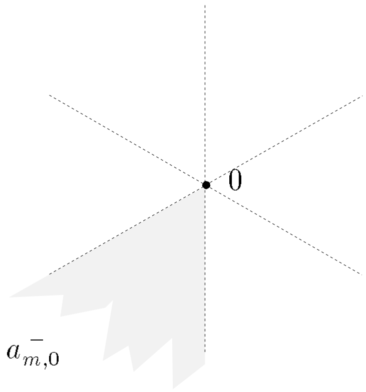


FIGURE 2

A careful definition of $\mathfrak{a}_{m,0}^-$ produces a natural order relation $\leq$ on $(\mathfrak{a}_m)^*$. The *leading asymptotic exponents* and the *leading homology exponents* are then defined by

$$\begin{aligned}
\mathcal{A}^0_{\text{asy}}(\pi) &= \{\text{minimal elements of } \mathcal{A}_{\text{asy}}(\pi) \\
&\qquad\qquad \text{with respect to the order } \leq\}, \text{ and} \\
\mathcal{A}^0_{\text{hom}}(\pi) &= \{\text{minimal elements of } \mathcal{A}_{\text{hom}}(\pi) \\
&\qquad\qquad \text{with respect to the order } \leq\},
\end{aligned}$$

respectively. Casselman's proof of (4) proceeds by showing: $0 \neq \mathcal{A}^0_{\text{asy}}(\pi) \subseteq \mathcal{A}^0_{\text{hom}}(\pi)$.

Although this discussion accurately describes the history of (4), this is not the approach in *Real reductive groups*.I. Rather, Wallach gives an ingenious inductive argument, depending upon Harish-Chandra's subquotient theorem plus some rough asymptotic estimates for certain special induced representations (the *spherical principal series*). This allows a proof of the subrepresentation theorem prior to a full blown discussion of matrix coefficients. Also, Beilinson and Bernstein have given a geometric proof of (4) using the theory of *D-modules*.

Fundamentally, the idea behind the classification is to inductively parametrize the irreducible admissible representations of a real reductive group G in terms of irreducible tempered representations of Levy subgroups L of G. We have seen that each irreducible admissible representation π occurs as a subrepresentation of some P_m-induced representation. With this in mind, a first approach toward the classification might be to consider the map

$$\psi : \widehat{G}_a \to \mathcal{P}_m \underset{\text{def}}{=} \{P_m - \text{ induced representations of } G\},$$

$\psi(\pi)$ denoting the P_m-induced representations into which we can embed a given irreducible representation π, using $\mathcal{A}^0_{\text{hom}}(\pi)$ and (3). There are two immediate problems with this approach. First of all, we may have $\#|\mathcal{A}^0_{\text{hom}}(\pi)| > 1$, so that ψ need not be single valued. Secondly, we may well have $\mathcal{A}^0_{\text{hom}}(\pi) = \mathcal{A}^0_{\text{hom}}(\pi')$, so that ψ need not be one to one. However, if we could make ψ injective and single valued by intersecting its image with some distinguished subset $\mathcal{P}'_m \subseteq \mathcal{P}_m$, then ψ would provide a classifying scheme; this works for *complex* reductive groups (like $Sl_n\mathbb{C}$), but the approach fails, in general. Nevertheless, we take these ideas as a model for the eventual classification and begin with a bit of philosophy:

(5) *The classification is a scheme to list representations according to the growth of their matrix coefficients, the idea being that tempered representations have nicest growth.*

The Langlands classification (Chapter 5) will organize nontempered representations according to growth by bringing various parabolic subgroups $P \neq P_m$ into the picture. The idea is to attach to each irreducible admissible representation (π, H_π) a set of data which will ultimately classify π, modulo the classification of irreducible tempered representations. This data, referred to as the *Langlands data* for π, is a triple

$$\mathcal{L}ang(\pi) = (P_\pi, \sigma_\pi, \nu_\pi), \qquad \text{where } P_\pi = M_\pi A_\pi N_\pi$$

is a parabolic subgroup of G, σ_π is an irreducible tempered representation of M_π and e^{ν_π} is a "negative" character on A_π, meaning that ν_π lies in a "negative cone" $(a^*_{\pi,0})^-$ analogous to the convex cone $(\mathfrak{a}_{m,0})^-$. Miličić's refinement of Langlands original argument shows that π is the unique irreducible subrepresentation of $\mathrm{Ind}_{P_\pi}(\sigma_\pi \otimes e^{\nu_\pi})$ and π occurs as a subquotient of $\mathrm{Ind}_{P_\pi}(\sigma_\pi \otimes e^{\nu_\pi})$ exactly once. Conversely, given a triple $\mathcal{L} = (P, \sigma, \nu)$ as above, called a collection of *Langlands data for* G, then $\mathrm{Ind}_P(\sigma \otimes e^{\nu})$ contains a unique irreducible subrepresentation π and $\mathcal{L}ang(\pi) = \mathcal{L}$. In summary

(6) **Langlands classification.** *There is a one-to-one correspondence between* $\widehat{G}_a$ *and the set of all Langlands data for* G.

The obvious question is how to associate $\mathcal{L}ang(\pi) = (P_\pi, \sigma_\pi, \nu_\pi)$ to a given irreducible admissible representation π of G and the answer will simultaneously define the notion of a tempered representation. Cheating slightly (and not for the first time), we view $\mathcal{A}^0_{\mathrm{hom}}(\pi)$ as a subset of $(\mathfrak{a}_{m,0})^*$. Given any $\nu \in \mathcal{A}^0_{\mathrm{hom}}(\pi)$, let $v^\#$ denote the element of closure $(\mathfrak{a}^*_{m,0})^-$ which is nearest ν; such a point will be uniquely defined, since closure $(a^*_{m,0})^-$ is a closed convex set in the Hilbert space $a^*_{m,0}$ (under the *Killing form* inner product). The *Langlands parameter* $\lambda(\pi)$ of π is defined to be a maximal length element of the set $\{\nu^\# : \nu \in \mathcal{A}^0_{\mathrm{hom}}(\pi)\}$. Three things can happen, as described below and illustrated in Figure 3.

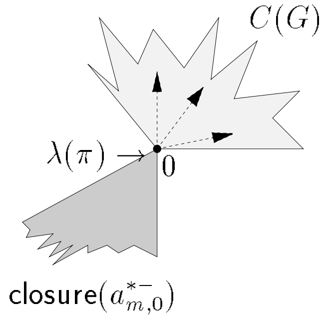
Case 1. $\lambda(\pi) = 0$. In this case, all vectors in $\mathcal{A}^0_{\mathrm{hom}}(\pi)$ lie inside the (closed) *tempered cone* $\mathcal{C}(G)$. By definition, any such π is called a *tempered representation of* G. In this case, the Langlands data of π is $(G, \pi, 0)$.

Case 2. $\lambda(\pi) \in (\mathfrak{a}^*_{m,0})^-$. In this case, some $\nu' \in \mathcal{A}^0_{\mathrm{hom}}(\pi)$ lies in the interior of closure $(\mathfrak{a}^*_{m,0})^-$ and we define $\lambda(\pi) = \nu'$. The Langlands data of π is (P_m, σ_m, ν'), where σ_m is determined by a subquotient of $H_0(n_m, \pi)$ of A_m representation type $e^{\nu'}$.

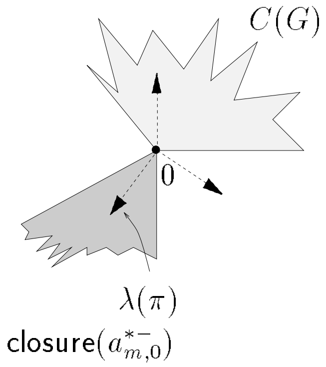
Case 3. $\lambda(\pi) \in$ boundary[closure $(\mathfrak{a}^*_{m,0})^-$]. In this case, one needs to utilize the structure theory of real reductive groups to see that the parameter $\lambda(\pi)$ determines a particular parabolic subgroup $P_\pi = M_\pi A_\pi N_\pi$ containing P_m and the parameter $\lambda(\pi)$ will lie inside $(\mathfrak{a}^*_{\pi,0})^-$. The Langlands data for π will be $(P_\pi, \sigma_\pi, \nu_\pi)$, where $\lambda(\pi) = \nu_\pi$ and σ_π is an irreducible subquotient of $H_0(\mathfrak{n}_\pi, \pi)$ of A_π representation type e^{ν_π}. The proof that this is indeed a collection of Langlands data involves showing that the exponents of σ_π lie in the tempered cone $\mathcal{C}(M_\pi)$; this depends upon relating our choice of $\lambda(\pi)$ to the convex cones $(\mathfrak{a}^*_{m,0})^-$ and $(\mathfrak{a}^*_{\pi,0})^-$.

The classification of tempered representations

With the Langlands classification behind us, a more detailed analysis of tempered growth behavior is required. The definition of a tempered representation was that all of the homology (or equivalently asymptotic) exponents lie inside the closed tempered cone $\mathcal{C}(G)$; recall Figure 3. This condition is equivalent to a growth estimate on the matrix coefficients called the *weak inequality* (too complicated to state in our exposition). One can contrast the temperedness condition on π with the stronger requirement that all of the homology (or equivalently asymptotic) exponents of π lie in the *interior* of the closed tempered cone $\mathcal{C}(G)$, in which case we say that π is *rapidly decreasing.* The rapidly decreasing condition is equivalent to a growth estimate on the matrix coefficients called the *strong inequality* (likewise complicated) and, in turn, this is equivalent to the square integrability of the matrix coefficients of π. In particular, rapidly decreasing representations are unitary.



Case 1



Case 2

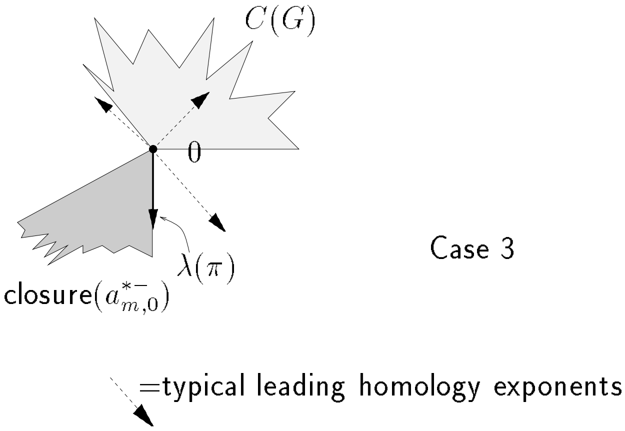


FIGURE 3

A result of Harish-Chandra, Langlands and Trombi asserts: Every irreducible tempered representation is a direct summand of a P-induced representation $I_P(\delta \otimes e^{i\mu})$, with δ square integrable and $\mu \in \mathfrak{a}_0^*$. This result says that every irreducible tempered representation is unitary, justifying one of our containments in Figure 1 (since our induction process takes unitary representations to unitary representations). Moreover, we now see that the classification is complete once we classify the square integrable representations of a real reductive group and unambiguously describe the summands of each such $I_P(\delta \otimes e^{i\mu})$.

The classification of the irreducible square integrable representations involves an *existence result* and an *exhaustion result*. The original approach, due to Harish-Chandra, involved a detailed analysis of the Fourier transform Θ_π for π an admissible representation of G; earlier, we only defined this for unitary representations, but the definition works in our wider context. Harish-Chandra made a detailed study of the singularities of the distribution Θ_π, one of the consequences of which was the following:

(7) **Regularity Theorem.** *The distribution Θ_π is given by integration against a locally integrable function on G (still denoted by Θ_π and called the character of π) which is real analytic on a full Haar measure dense subset G' of G.*

Using this, Harish-Chandra proceeded to construct the characters of the irreducible square integrable representations.

By contrast, Wallach's development (Chapter 6) is to postpone the regularity theorem and construct the candidates for the square integrable representations directly. This construction involves an induction procedure different than the one discussed earlier. This process, often called *cohomological parabolic induction*, is a homological construction of representations, suggested by work of Gregg Zuckerman in the mid 1970s. As with our previous notion of induction, cohomological parabolic induction starts with a Harish-Chandra module of a real reductive subgroup L of G and constructs a Harish-Chandra module for G. The idea comes from complex geometry and roughly amounts to algebraicizing *Dolbeault cohomology groups* to obtain a family of *derived functors* $\mathcal{R}_q^k : \mathcal{HC}_L \to \mathcal{H}C_G$, see [5]. The index q refers to a subalgebra of $\mathfrak{g}$ containing a fixed maximal solvable subalgebra and, additionally, when we decompose $q = l \oplus u$, we want l to be the complexified Lie algebra of a real reductive subgroup L of G, but u need not have this property. In fact, if we apply the $\mathcal{R}_q^k$ construction in the case when q is the complexified Lie algebra of a parabolic subgroup Q of G, then we obtain the earlier notion of induction.

If we begin with suitable choices of q, s and ξ (a representation of L), then $\mathcal{R}_q^s(\xi)$ are irreducible unitary representations of G, defined to be the *discrete series representations*. The proof of the unitarity of these modules in the book is taken from a paper of the author [6]. Further asymptotic analysis then shows that the discrete series representations are rapidly decreasing, hence square integrable representations. To classify the square integrable representations, it suffices to show that every square integrable representation is a discrete series. This involves the introduction of a function space, the space of *cusp forms* (Chapter 7), which contains the matrix coefficients of square integrable representations; in fact, it is spanned by them. In particular, the matrix coefficients of the discrete series lie in the space of cusp forms. Chapter 8 then builds on these ideas to prove that the class of discrete series representations exhausts the class of all irreducible square integrable representations, referred to as the *Completeness Theorem* of Harish-Chandra.

To complete the classification of tempered representations, we need to describe the summands of $I_P(\delta \otimes e^{i\mu})$, with δ square integrable and $\mu \in \mathfrak{a}_0^*$. This was carried out by Knapp-Zuckerman [4] and involves (among other things) a *deformation theory* for representations. This theory, commonly called *coherent continuation*, depends on a discrete parameter and allows us to go from the discrete series to *limits of discrete series*, which are the parametrizing building blocks in the Knapp-Zuckerman classification. This completes the classification of $\widehat{G}_a$.

Two related topics

We have indicated a close connection between the asymptotic behavior of matrix coefficients and the $\mathfrak{n}$-homology groups of π. In turn, these notions are very closely related to the distributions Θ_π, viewed as functions on G'. One is led to the set of *leading character exponents* of π on A_m^-, denoted $\mathcal{A}^0_{\text{char}}$, and this set coincides with the previously defined sets $\mathcal{A}^0_{\text{hom}}$ and $\mathcal{A}^0_{\text{asy}}$. In fact, one can show that Θ_π restricted to $G' \cap M_m A_m^-$ can be expressed as a quotient of two functions, the numerator of which is the *Euler characteristic* of $H_*(\mathfrak{n}_m, \pi)$ as a $M_m A_m$ representation. This result, the *Osborne conjecture*, was proved by Hecht-Schmid in the late 1970s and ties together the theory of characters, asymptotics, and $\mathfrak{n}$-homology presented in Wallach's book.

A second topic of related interest is the *algebraic character theory* of Harish-Chandra modules. This is one objective of Vogan's book *Representations of real reductive Lie groups* [5]. By the regularity theorem, we know that Θ_π can be represented by a function on G' and one can pose the problem:

(8) **Character Problem.** *Compute the characters of the irreducible admissible representations.*

To do so, one shows that it is enough to compute the restriction of Θ_π to certain commutative subgroups of G, called *Cartan subgroups*. (This idea of restricting to commutative subgroups illustrates a fundamental philosophy throughout all of Harish-Chandra's work: Reformulate the problem at hand in terms of harmonic analysis of commutative subgroups of G. The "character Θ_π" is so named, since its expression involves a finite combination of classical one-dimensional characters of these Cartan subgroups.) The character $\Theta_{\text{Ind}_P(\delta\otimes e^\nu)}$ of a P-induced representation $\text{Ind}_P(\delta \otimes e^\nu)$, with δ an irreducible tempered representation of L, can be computed via an algorithm of Harish-Chandra; we will call these *good characters*. Then, given an irreducible admissible representation π of G and working within an appropriate *Grothendieck group*, one can express

$$\Theta_\pi = \sum_{\text{good characters}} c_{P,\delta\nu} \Theta_{\text{Ind}_P(\delta\otimes e^\nu)}, \qquad c_{P,\delta,\nu} \in Z.$$

To solve the character problem, one must compute the integers $c_{P,\delta,\nu}$. Vogan's book [5] formulates a conjecture to compute the numbers $c_{P,\delta,\nu}$, which he has since proved, thus solving the character problem. This solution depends upon connections between *Representation theory*, *D-module theory* and *Intersection cohomology*.

Final remarks

Another recent book which covers some of the same ground, but from a different viewpoint is Knapp's *Representation theory of semisimple groups* (3). It differs in at least three ways: Wallach supplies complete proofs (and extensive appendices), whereas Knapp covers additional ground by sketching proofs or, in some cases, proving results in special cases. Also, Knapp's approach is more analytic, as exemplified by the different treatment of the discrete series. Finally, Knapp's book contains a large supply of examples, valuable to both novice and expert. Because of these differences, any serious reader of *Real reductive groups.*I should have Knapp's book [3], together with Vogan's book [5], close at hand.

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