Quadratic Convergence of SQP-Like Methods for Convex-Composite Optimization

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Convex-Composite Optimization

\[ \min_{x \in \mathbb{R}^n} f(x) := h(c(x)) \]  \hspace{1cm} (P)

\[ h : \mathbb{R}^m \to \mathbb{R} \cup \{ +\infty \} \text{ is closed, proper, convex} \]

\[ c : \mathbb{R}^n \to \mathbb{R}^m \text{ is } C^2\text{-smooth} \]
Convex-Composite Optimization

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The Model

The Data
Convex-Composite Optimization

\[
\min_{x \in \mathbb{R}^n} f(x) := h(c(x)) + g(x) \quad (P)
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- \(h : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}\) is closed, proper, convex \quad The Model
- \(c : \mathbb{R}^n \to \mathbb{R}^m\) is \(C^2\)-smooth \quad The Data
- \(g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}\) is closed, proper, convex \quad Regularization
  used to induce solution properties
Convex-Composite Optimization

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\min_{x \in \mathbb{R}^n} f(x) := h(c(x)) + g(x) \quad (P)
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Regularization used to induce solution properties

70’s
Fletcher, Powel, Osborne

80-90’s
Burke, Ferris, Fletcher, Kawasaki, Masden, Poliquin, Powel, Osborne, Rockafellar, Womersley, Wright, Yuan

Recent (15-19’s)
Aravkin, Bell, B, Chang, Cui, Duchi, Davis, Drusvyatskiy, Hoheisel, Hong, Lewis, Ioffe, Mordukhovich, Pang, Ruan
Examples: 70 - 90’s

Non-linear least-squares: \( f(x) = \| c(x) \|^2 \)
Examples: 70 - 90’s

Non-linear least-squares: $f(x) = \|c(x)\|_2^2$

**Feasibility:** $c(x) \in C : \min \text{dist} (c(x) \mid C)$,
where $C \subset \mathbb{R}^m$ is non-empty, closed, convex, and
$\text{dist} (y \mid C) := \inf \{ \|y - z\| \mid z \in C \}$. 
Examples: 70 - 90’s

Non-linear least-squares: \( f(x) = ||c(x)||^2_2 \)

**Feasibility:** \( c(x) \in C : \min \text{dist} (c(x) \mid C) \),
where \( C \subset \mathbb{R}^m \) is non-empty, closed, convex, and
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\text{dist} \ (y \mid C) := \inf \ \{ ||y - z|| \mid z \in C \}.
\]

**Exact Penalization:** \( \min \varphi(x) + \alpha \text{dist} (\hat{c}(x) \mid C) \)
Here \( c(x) := (\varphi(x), \hat{c}(x)) \) and \( h(\mu, y) := \mu + \alpha \text{dist} (y \mid C) \)
Examples: 70 - 90’s

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Here \( c(x) := (\varphi(x), \hat{c}(x)) \) and \( h(\mu, y) := \mu + \alpha \text{dist} (y | C) \)

Non-linear programming: \( \min \varphi(x) + \delta_C(\hat{c}(x)) \).
Here \( c(x) := (\varphi(x), \hat{c}(x)) \) and \( h(\mu, y) := \mu + \delta_C(y) \), where
\( \delta_C(y) = 0 \) if \( y \in C \) and \(+\infty\) otherwise.
Optimal Value Composition:

\[ h(c) := \min \left\{ b^\top y \mid Ay \leq c \right\} \]
More Recent Examples

Optimal Value Composition:

\[ h(c) := \min \left\{ b^\top y \mid Ay \leq c \right\} \]

Quadratic support functions:

\[ h(c) := \sup_{u \in U} \langle u, Bc \rangle - \frac{1}{2} u^T M u \]

with \( U \subset \mathbb{R}^k \) non-empty, closed, convex, \( M \in \mathbb{S}^n \) is positive semi-definite.
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Piecewise linear-quadratic (PLQ) penalties: (Rockfellar-Wets (97))

Quadratic support functions with \( U \subset \mathbb{R}^k \) non-empty, closed and convex polyhedron.
Dual representation of PLQs

\[ \frac{1}{2} x^2 = \sup_{u \in \mathbb{R}} \langle u, x \rangle - \frac{1}{2} u^2 \]

\[ Q_{0.8}(x) = \sup_{u \in [-0.8, 0.2]} \langle u, x \rangle \]

\[ \rho_h(x) = \sup_{u \in [-\kappa, \kappa]} \langle u, x \rangle - \frac{1}{2} u^2 \]
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\]

PLQ penalties closed under addition and affine composition.
## PLQ penalties in practice

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The Convex-Composite Lagrangian

\[ \textbf{P} \min_{x \in \mathbb{R}^n} h(c(x)) \]

- The Lagrangian for \( \textbf{P} \): (B. (87))

\[ L(x, y) := \langle y, c(x) \rangle - h^*(y) \]

- The conjugate of \( h \) given by the support function for \( \text{epi}(h) \),

\[ h^*(y) := \sup_{x} [\langle y, x \rangle - h(x)] \]
The Convex-Composite Lagrangian

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\begin{cases}
\text{(Primal)} & \inf_x \sup_y L(x, y) \\
\text{(Dual)} & \sup_y \inf_x L(x, y)
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- The conjugate of \( h \) given by the support function for \( \text{epi}(h) \),

\[
h^*(y) := \sup_{x}[ \langle y, x \rangle - h(x) ] = \sup_{(x, \mu) \in \text{epi}(h)} \langle (y, -1), (x, \mu) \rangle
\]
\( \mathbf{P}_k \) \( \min_x h \left( c(x^k) + \nabla c(x^k)[x - x^k] \right) + \frac{1}{2} (x - x^k)^\top H_k (x - x^k), \)

- \( H_k \) approximates the Hessian of a Lagrangian for \( \mathbf{P} \) at \((x^k, y^k)\)
- Newton’s method: \( H_k := \nabla_{xx}^2 L(x^k, y^k) = \sum_{k=1}^m y_i^k \nabla_{xx}^2 c_i(x^k) \)
- \( \mathbf{P}_k \) may or may not be convex depending on whether \( H_k \succeq 0 \).
- A example is the Gauss-Newton method: \( h = \| \cdot \|_2^2 \)
  \[ \min_x \| c(x^k) + c'(x^k)(x - x^k) \|_2^2 \]
Algorithm for NLP

\[ \text{NLP} \quad \text{minimize} \quad \phi(x) \]
subject to \( f_i(x) = 0, \ i = 1, \ldots, s, \ f_i(x) \leq 0, \ i = s+1, \ldots, m. \)
Algorithm for NLP

NLP minimize $\phi(x)$
subject to $f_i(x) = 0, \ i = 1, \ldots, s, \ f_i(x) \leq 0, \ i = s+1, \ldots, m.$

• Convex-Composite Framework

$$h(\mu, y) = \mu + \delta_K(y),$$

$$c(x) = (\phi(x), f(x))$$

$$L(x, y) = \phi(x) + \sum_{k=1}^{m} y_i f_i(x) - \delta_{K^\circ}(y), \quad K^\circ = \mathbb{R}^s \times \mathbb{R}^{m-s}$$
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- Subproblems:

\[ \mathbf{P}_k \text{ minimize } \phi(x^k) + \nabla \phi(x^k)^T (x - x^k) + \frac{1}{2} [x - x^k]^T H_k [x - x^k] \]
subject to $f_i(x^k) + \nabla f_i(x^k)^T (x - x^k) = 0$, $i = 1, \ldots, s$
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\begin{align*}
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    c(x) &= (\phi(x), f(x)) \\
    L(x, y) &= \phi(x) + \sum_{k=1}^{m} y_i f_i(x) - \delta_{K^\circ}(y),
\end{align*}
\]

\( K := \{0\}^s \times \mathbb{R}^{m-s} \)

\( K^\circ = \mathbb{R}^s \times \mathbb{R}^{m-s} \)

• Subproblems: Sequential quadratic programming (SQP)

\[
\begin{align*}
    P_k & \quad \text{minimize} \quad \phi(x^k) + \nabla \phi(x^k)^T (x - x^k) + \frac{1}{2} [x - x^k]^\top H_k [x - x^k] \\
    \text{subject to} \quad f_i(x^k) + \nabla f_i(x^k)^T (x - x^k) = 0, \ i = 1, \ldots, s \\
    f_i(x^k) + \nabla f_i(x^k)^T (x - x^k) = 0, \ i = s + 1, \ldots, m.
\end{align*}
\]
Convergence of Convex-Composite Newton’s Method

Robinson (72):
Assumed \( h = \delta_K \) with \( K := \{0\}^s \times \mathbb{R}_m^{m-s} \) (NLP case).

Established quadratic convergence in the NLP case under linear independence of the active constraint gradients, strict complementarity, and strong second-order sufficiency.

Robinson (80):

Introduced the revolutionary notion of generalized equations which, among many other consequences, re-established quadratic convergence for NLP. The generalized equations approach is much more powerful as it allows access to a very rich sensitivity theory including metric regularity properties of solution mappings.
Convergence of Convex-Composite Newton’s Method

**Womersley (85):**
Assumed $h$ is finite-valued piecewise linear convex.

*Established quadratic convergence under NLP-like conditions: linear independence of the active constraint gradients, strict complementarity, and strong second-order sufficiency.*

**B-Ferris (95):**
Assumed $h$ is finite-valued closed, proper, convex.

*Established quadratic convergence when $C := \text{arg min } h$ is a set of weak sharp minima for $h$, and $\text{arg min } f = \{x \mid c(x) \in C\}$.***

**Cibulka-Dontchev-Kruger (16):**
Assumed $h$ is piecewise linear convex.

*Established super-linear convergence under the Dennis-Moré conditions using generalized equations.*
A long standing open problem:

Can one establish second-order rates using the rich history of second-order ideas for convex-composite functions?

(B(87), Kawazaki(88), Ioffe(88), B-Poliquin(92), Rochafellar-Wets(92), Nguyen(17-19))
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Solution Proposal:

Develop a generalized equations approach for the PLQ class using PLQ second-order theory and partial smoothness to establish second-order rates under hypotheses motivated by those used for NLP.
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Solution Proposal:

*Develop a generalized equations approach for the PLQ class using PLQ second-order theory and partial smoothness to establish second-order rates under hypotheses motivated by those used for NLP.*

Key new ingredient is *partial smoothness* due to (Lewis (02)).
PLQ Functions

$h : \mathbb{R}^m \to \overline{\mathbb{R}}$ is called piecewise linear-quadratic (PLQ) if $\text{dom } h \neq \emptyset$ and, for $\mathcal{K} \geq 1$,

$$\text{dom } h = \bigcup_{k=1}^{\mathcal{K}} C_k,$$

where the sets $C_k$ are convex polyhedrons,

$$C_k = \{ c \mid \langle a_{kj}, c \rangle \leq \alpha_{kj}, \text{ for all } j \in \{1, \ldots, s_k\} \},$$

and relative to which $h(c)$ is given by an expression of the form

$$h(c) = \frac{1}{2} \langle c, Q_k c \rangle + \langle b_k, c \rangle + \beta_k \quad \forall c \in C_k$$

with $\beta_k \in \mathbb{R}$, $b_k \in \mathbb{R}^n$, and $Q_k \in \mathbb{S}^m$. 
Variational Analysis of PLQ-Composite Functions

Assume $f := h \circ c$ with $h$ convex PLQ and $c$ in $C^2(\mathbb{R}^n)$.

**Active Set:** For $c \in \text{dom } h$, the active set at $c$ is
$$\mathcal{K}(c) := \{k \mid c \in C_k\}.$$  

**Basic Constraint Qualification:** (BCQ)
$$\ker \nabla c(\bar{x})^\top \cap N_{\text{dom } h}(c(\bar{x})) = \{0\}$$

**Subdifferential:** Under the BCQ,
$$\partial f(x) = c'(x)^T \partial h(c(x)).$$

**Directional Derivative:** Under BCQ
$$f'(x; d) = \lim_{t \downarrow 0} \frac{f(x+td)-f(x)}{t} = h'(c(x); c'(x)d)$$
with
$$h'(\bar{c}; w) = \langle Q_k \bar{c} + b_k, w \rangle \quad \forall \ k \in \mathcal{K}(\bar{c}) \text{ and } w \in T_{C_k}(\bar{c}) .$$
Directions of Non-Ascent and Multipliers

Directions of non-ascent:

\[ D(x) := \{ d \in \mathbb{R}^n \mid f'(x : d) \leq 0 \} \]
\[ = \{ d \in \mathbb{R}^n \mid h'(c(x); \nabla c(x)d) \leq 0 \} \]  \hskip 1cm (BCQ)

The Multiplier Set:

\[ M(\bar{x}) := \ker \nabla c(\bar{x})^\top \cap \partial h(c(\bar{x})) = \left\{ y \mid \begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \begin{pmatrix} \partial_x L(\bar{x}, y) \\ \partial_y (-L)(\bar{x}, y) \end{pmatrix} \right\} \]
The Second Directional Derivative

The PLQ second directional derivative:
(Rockafellar-Wets (97))

\[ 0 \leq h''(\bar{c}; w) := \lim_{t \searrow 0} \frac{h(\bar{c} + tw) - h(\bar{c}) - th'(\bar{c}; w)}{\frac{1}{2} t^2} \]

\[ = \begin{cases} 
\langle w, Q_k w \rangle & \text{when } w \in T_{C_k}(\bar{c}), \\
\infty & \text{when } w \notin T_{\text{dom } h(\bar{c})}.
\end{cases} \]

and \( h''(\bar{c}; \cdot) \) is PLQ, but not necessarily convex.
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\end{cases} \]

and \( h''(\bar{c}; \cdot) \) is PLQ, but not necessarily convex.

Moreover, there exists a neighborhood \( V \) of \( \bar{c} \) such that

\[ h(c) = h(\bar{c}) + h'(\bar{c}; c - \bar{c}) + \frac{1}{2} h''(\bar{c}; c - \bar{c}) \text{ for } c \in V \cap \text{dom } h. \]
PLQ-Composite 2nd-Order Nec. and Suff. Conditions

(Rockafellar-Wets (97))
Let \( \bar{x} \in \text{dom} \, f \) such that \( f \) satisfies BCQ at \( \bar{x} \).

(1) (Nec.) If \( f \) has a local minimum at \( \bar{x} \), then
\[
0 \in \nabla c(\bar{x})^\top \partial h(c(\bar{x})) \quad \text{and,} \quad \forall \, d \in D(\bar{x}),
\]
\[
h''(c(\bar{x}); \nabla c(\bar{x})d) + \max \{ \langle d, \nabla^2_{xx} L(\bar{x}, y)d \rangle \mid y \in M(\bar{x}) \} \geq 0.
\]

(2) (Suff.) If \( 0 \in \nabla c(\bar{x})^\top \partial h(c(\bar{x})) \) and, \( \forall \, d \in D(\bar{x}) \setminus \{0\}, \)
\[
h''(c(\bar{x}); \nabla c(\bar{x})d) + \max \{ \langle d, \nabla^2_{xx} L(\bar{x}, y)d \rangle \mid y \in M(\bar{x}) \} > 0,
\]
then \( \bar{x} \) is a strong local minimizer of \( f \),
that is, there exists \( \varepsilon > 0, \mu > 0 \) such that
\[
f(x) \geq f(\bar{x}) + \frac{\mu}{2} \|x - \bar{x}\|_2^2 \quad \forall \, x \in B(\bar{x}, \varepsilon).
\]
Convex-Composite Generalized Equations

Let \( f := h \circ c \) be convex-composite, and define the set-valued mapping \( g + G : \mathbb{R}^{n+m} \Rightarrow \mathbb{R}^{n+m} \) by

\[
g(x, y) = \begin{pmatrix} \nabla c(x) \top y \\ -c(x) \end{pmatrix}, \quad G(x, y) = \begin{pmatrix} \{0\}^n \\ \partial h^*(y) \end{pmatrix}.
\]

The associated generalized equation for \( P \) is \( g + G \ni 0 \).
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$$

The associated generalized equation for $P$ is $g + G \ni 0$.

For a fixed $(\bar{x}, \bar{y}) \in \mathbb{R}^n \times \mathbb{R}^m$, define the linearization mapping

$$
\mathcal{G} : (x, y) \mapsto g(\bar{x}, \bar{y}) + \nabla g(\bar{x}, \bar{y}) \begin{pmatrix} x - \bar{x} \\ y - \bar{y} \end{pmatrix} + G(x, y),
$$

where $\nabla g(\bar{x}, \bar{y}) = \begin{pmatrix} \nabla^2(\bar{y}c)(\bar{x}) & \nabla c(\bar{x})^\top \\ -\nabla c(\bar{x}) & 0 \end{pmatrix}$.
Newton’s Method for Generalized Equations

- Let $f := h \circ c$ be convex-composite.
- For $(\hat{x}, \hat{y}) \in \mathbb{R}^n \times \mathbb{R}^m$ set $\hat{H} := \nabla_{xx}^2 L(\hat{x}, \hat{y})$.
- Assume $f$ satisfies BCQ at $\hat{x}$.

Then, $(\tilde{x}, \tilde{y}) \in \mathbb{R}^n \times \mathbb{R}^m$ satisfy the optimality conditions for

$$\min_{x \in \mathbb{R}^n} h(c(\hat{x})) + \nabla c(\hat{x})(x - \hat{x}) + \frac{1}{2}(x - \hat{x})^\top \hat{H}(x - \hat{x})$$

if and only if $(\tilde{x}, \tilde{y})$ solves the Newton equations for $g + G$:

$$0 \in g(\hat{x}, \hat{y}) + \nabla g(\hat{x}, \hat{y}) \begin{pmatrix} x - \hat{x} \\ y - \hat{y} \end{pmatrix} + G(x, y).$$
Strong Metric Subregularity

A set-valued mapping $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is strongly metrically subregular at $\bar{u}$ for $\bar{v}$ if $(\bar{u}, \bar{v}) \in \text{graph}(S)$ and there exists $\kappa \geq 0$ and a neighborhood $U$ of $\bar{u}$ such that

$$\|u - \bar{u}\| \leq \kappa \text{dist}(\bar{v} \mid S(u))$$

for all $u \in U$. 

Theorem: (B-Engel(18))

$h : \mathbb{R}^m \rightarrow \mathbb{R}$ convex PLQ and $f := h \circ c$ satisfies BCQ at $\bar{x} \in \text{dom} f$. Then, the following are equivalent:

(1) The multiplier set $M(\bar{x}) := \text{ker} \nabla c(\bar{x})^\top \cap \partial h(c(\bar{x}))$ is a singleton $\{\bar{y}\}$ and the second-order sufficient conditions are satisfied at $\bar{x}$.

(2) The mapping $g + G$ is strongly metrically subregular at $(\bar{x}, \bar{y})$ for 0 and $\bar{x}$ is a strong local minimizer of $f$. 

Corollary: The matrix secant method converges superlinearly if the Dennis-More condition holds.
Strong Metric Subregularity

A set-valued mapping \( S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m \) is \textit{strongly metrically subregular} at \( \bar{u} \) for \( \bar{v} \) if \( (\bar{u}, \bar{v}) \in \text{graph}(S) \) and there exists \( \kappa \geq 0 \) and a neighborhood \( U \) of \( \bar{u} \) such that
\[
\|u - \bar{u}\| \leq \kappa \text{dist}(\bar{v} \mid S(u)) \quad \text{for all} \quad u \in U.
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**Theorem:** (B-Engel(18)) \( h : \mathbb{R}^m \rightarrow \mathbb{R} \) convex PLQ and \( f := h \circ c \) satisfies BCQ at \( \bar{x} \in \text{dom} \, f \). Then, the following are equivalent:

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**Theorem:** (B-Engel(18)) $h : \mathbb{R}^m \to \overline{\mathbb{R}}$ convex PLQ and $f := h \circ c$ satisfies BCQ at $\bar{x} \in \text{dom} f$. Then, the following are equivalent:

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**Corollary:** The matrix secant method converges superlinearly if the Dennis-Móre condition holds.
Partial Smoothness: Lewis (02)

- $h : \mathbb{R}^m \to \overline{\mathbb{R}}$ is a closed and proper function.
- $\mathcal{M}$ a $C^2$-smooth manifold and $\bar{c} \in \mathcal{M} \subset \mathbb{R}^m$.

The function $h$ is *partly smooth* at $\bar{c}$ relative to $\mathcal{M}$ if $h$ the following four properties hold:

1. (restricted smoothness) the restriction $h|_{\mathcal{M}}$ is smooth around $\bar{c}$, in that there exists a neighborhood $V$ of $\bar{c}$ and a $C^2$-smooth function $g$ defined on $V$ such that $h = g$ on $V \cap \mathcal{M}$;
2. (existence of subgradients) at every point $c \in \mathcal{M}$ close to $\bar{c}$, $\partial h(c) \neq \emptyset$;
3. (normals and subgradients parallel) $\text{par}\partial h(\bar{c}) = N_{\mathcal{M}}(\bar{c})$;
4. (subgradient inner semicontinuity) the subdifferential map $\partial h$ is inner semicontinuous at $\bar{c}$ relative to $\mathcal{M}$.

Generalizes classical notions of nondegeneracy, strict complementarity, and active constraint identification.
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Partial Smoothness
Rockafellar-Wets Representation (RWR)

$h$ is PLQ and $\text{int} \left( \text{dom} \, h \right) \neq \emptyset$. Then, WLOG, the polyhedral sets $\{C_k\}_{k=1}^K$ are given in terms of a common set of $s > 0$ hyperplanes $H := \{(a_j, \alpha_j)\}_{j=1}^s \subset (\mathbb{R}^m \setminus \{0\}) \times \mathbb{R}$, so that $\forall \, k \in \{1, \ldots, K\}$,

$$C_k = \left\{ c \mid \langle \omega_{kj} a_j, c \rangle \leq \omega_{kj} \alpha_j, \, \text{for all} \, j \in \{1, \ldots, s\} \right\},$$

with $\omega_{kj} \in \{\pm 1\}$,

$$I_k(c) = \{ j \mid \langle \omega_{kj} a_j, c \rangle = \omega_{kj} \alpha_j \} = \{ j \mid \langle a_j, c \rangle = \alpha_j \} \subset \{1, \ldots, s\},$$

and

(i) $\emptyset \neq \text{int} \left( C_k \right) = \left\{ c \mid \langle \omega_{kj} a_j, c \rangle < \omega_{kj} \alpha_j, \, \forall \, j \in \{1, \ldots, s_k\} \right\}, \, \forall \, k \in \{1, \ldots, K\},$

(ii) $\text{int} \left( C_{k_1} \right) \cap \text{int} \left( C_{k_2} \right) = \emptyset$ when $k_1 \neq k_2$.

Condition (b) implies that if $c \in C_{k_1} \cap C_{k_2}$, then $c \in \text{bdry} \, C_{k_1} \cap \text{bdry} \, C_{k_2}$ when $k_1 \neq k_2$. 
The Active Manifold

- $\mathcal{M}$ Active set: $\mathcal{K}(c) := \{ k \in \mathbb{R}^m \mid c \in C_k, \ k \in \{1, 2, \ldots, \mathcal{K}\}\}$

- Active Manifold: $\mathcal{M}_{\bar{c}} := \text{ri} \bigcap_{k \in \mathcal{K}(\bar{c})} C_k$

- Active set (RWR) for $C_k = \{c \mid \langle \omega_{kj} a_j, c \rangle \leq \omega_{kj} \alpha_j, \text{ for all } j \in \{1, \ldots, s\}\}$, with $\omega_{kj} \in \{\pm 1\}$, is

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The Active Manifold

**Lemma:** Let $\bar{c} \in \text{dom } f$ and assume $\text{dom } h$ is given by an RWR. Then, for all $c \in \mathcal{M}_{\bar{c}}$ and $k \in \mathcal{K}(\bar{c})$,

$$\mathcal{K}(c) = \mathcal{K}(\bar{c}), \quad \mathcal{M}_c = \mathcal{M}_{\bar{c}} \quad \text{and} \quad I_k(c) = I_k(\bar{c}).$$

Moreover,

$$\mathcal{M}_{\bar{c}} = \left\{ c \ \bigg| \begin{array}{l}
\langle c, a_j \rangle = \alpha_j \ \text{for all } k \in \mathcal{K}(\bar{c}), \ j \in I_k(\bar{c}) \\
\langle c, \omega_{kj} a_j \rangle < \omega_{kj} \alpha_j \ \text{for all } k \in \mathcal{K}(\bar{c}), \ j \notin I_k(\bar{c})
\end{array} \right\}$$
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$$

Moreover,

$$\mathcal{M}_{\bar{c}} = \left\{ c \mid \begin{array}{l}
\langle c, a_j \rangle = \alpha_j \text{ for all } k \in \mathcal{K}(\bar{c}), j \in I_k(\bar{c}) \\
\langle c, \omega_{kj} a_j \rangle < \omega_{kj} \alpha_j \text{ for all } k \in \mathcal{K}(\bar{c}), j \notin I_k(\bar{c})
\end{array} \right\}
$$

For $k \in \mathcal{M}_{\bar{c}}$ set $A := A_k(\bar{c})$ whose columns are

$$\{ a_j \mid k \in \mathcal{K}(\bar{c}), j \in I_k(\bar{c}) \}.$$

Then $\exists$ diagonal $P_j$ with entries $\pm 1$ on the diagonal such that

$$AP_j = A_{kj}(c) \quad \forall c \in \mathcal{M}_{\bar{c}},
$$

and, for any $k \in \mathcal{K}(\bar{c})$ and $c \in \mathcal{M}_{\bar{c}}$,

$$T_{\mathcal{M}_{\bar{c}}}(c) = \ker A^\top, \quad \text{and } N_{\mathcal{M}_{\bar{c}}}(c) = \text{Ran}(A).$$
The Subdifferential of \( h \)

We let \( \bar{k} = |\mathcal{K}(\bar{c}) | \) and \( \ell := |I_k(\bar{c})| = |I_{k'}(\bar{c})| \) for all \( k, k' \in \mathcal{K}(\bar{c}) \), so that \( A \in \mathbb{R}^{m \times \ell}, P_j \in \mathbb{R}^{\ell \times \ell}, P_{\bar{k}} = I_\ell \), and define block matrices

\[
\hat{Q} := \text{diag}(Q_k), \hat{A} := \text{diag}AP_j
\]

\[
\mathcal{A} := \begin{pmatrix}
(1 - \bar{k})AP_1 & AP_2 & \cdots & A \\
AP_1 & (1 - \bar{k})AP_2 & \cdots & A \\
\vdots & \ddots & \ddots & \vdots \\
AP_1 & AP_2 & \cdots & (1 - \bar{k})A
\end{pmatrix},
\]

\[
Q := \begin{bmatrix}
Q_{k_1} \\
Q_{k_2} \\
\vdots \\
Q_{\bar{k}}
\end{bmatrix}, \quad B := \begin{bmatrix}
b_{k_1} \\
b_{k_2} \\
\vdots \\
b_{\bar{k}}
\end{bmatrix}, \quad J := \begin{bmatrix}
I_m \\
I_m \\
\vdots \\
I_m
\end{bmatrix}
\]

and averaged quantities

\[
\bar{Q} = (1/\bar{k})J^\top \hat{Q}J, \quad \bar{A} = (1/\bar{k})J^\top \hat{A}, \quad \bar{b} = (1/\bar{k})J^\top B, \quad \lambda_0(\bar{c}) = \bar{Q}\bar{c} + \bar{b}.
\]
The Subdifferential of $h$

For any $c \in \mathcal{M}$, $\partial h(c)$ can be given by two equivalent formulations:

$$
\partial h(c) = \left\{ y \left| \begin{array}{c}
\exists \mu = (\mu_1^\top, \ldots, \mu_k^\top)^\top \geq 0 \\
\text{such that } Jy = Qc + B + \hat{A}\mu
\end{array} \right. \right\} = \lambda_0(c) + \bar{A}\mathcal{U}(c),
$$

where

$$
\mathcal{U}(c) := \left\{ \mu \geq 0 \mid A\mu = \bar{k} \left[ Qc + B - J(Qc + \bar{b}) \right] \right\}.
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Structure Functional of Osborne (01)
The Subdifferential of $h$

For any $c \in M_{\bar{c}}$, $\partial h(c)$ can be given by two equivalent formulations:

$$\partial h(c) = \left\{ y \mid \exists \mu = (\mu_1^T, \ldots, \mu_{\bar{k}}^T)^T \geq 0 \text{ such that } Jy = Qc + B + \hat{A}\mu \right\} = \lambda_0(c) + \bar{A}U(c),$$

where

$$U(c) := \left\{ \mu \geq 0 \mid A\mu = \bar{k} [Qc + B - J(Q\bar{c} + \bar{b})] \right\}.$$

**Nondegeneracy:** We say $M_{\bar{c}}$ satisfies the nondegeneracy condition if $\ker(A) = \{0\}$. 
The Subdifferential of $h$

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$$\left. \left. \text{such that } Jy = Qc + B + \hat{A} \mu \right\} = \lambda_0(c) + \bar{A} \mathcal{U}(c), \right.$$ 

where

$$\mathcal{U}(c) := \left\{ \mu \geq 0 \left| \mathcal{A} \mu = k \left[ Qc + B - J(\bar{Q}c + \bar{b}) \right] \right. \right\}.$$ 

**Nondegeneracy:** We say $\mathcal{M}_{\bar{c}}$ satisfies the nondegeneracy condition if $\ker(A) = \{0\}$.

**Lemma:** Let $c \in \mathcal{M}_{\bar{c}}$. If $\ker A = \{0\}$, then, for every $y \in \partial h(c)$, there is a unique $\mu(c, y) \in \mathcal{U}(c)$ such that $y = \lambda_0(c) + \bar{A} \mu(c, y)$. 
\[ \text{\textit{k-Strict Complementarity}} \]

Let \( \bar{c} \in \text{dom } h \). We say \textit{k-strict complementarity} holds at \( (c, y) \in \text{graph } (\partial h) \) for \( \mu = (\mu_1^\top, \ldots, \mu_k^\top)^\top \in \mathcal{U}(c) \) wrt \( \mathcal{M}_{\bar{c}} \) if

1. \( c \in \mathcal{M}_{\bar{c}} \) and \( y = \lambda_0(c) + \bar{A}\mu \),
2. \( \exists \ k \in \mathcal{K}(\bar{c}) \) with \( \mu_k > 0 \),
3. if \( j \in \mathcal{K}(c) \setminus \{k\} \) and \( i \in \{1, \ldots, \ell\} \) with \( (\mu_j)_i = 0 \), then the scalars \( (P_{jj'})_{ii} = 1 \) for all \( j' \in \mathcal{K}(c) \).
**k-Strict Complementarity**

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**Lemma:** Let $\bar{c} \in \text{dom } h$. If $\mathcal{M}_{\bar{c}}$ is nondegenerate and for some $c \in \mathcal{M}_{\bar{c}}$ and there is a $(c, y) \in \text{graph } (\partial h)$ such that k-strict complementarity holds at $(c, y)$ wrt $\mathcal{M}_{\bar{c}}$, then $\mathcal{M}_{\bar{c}}$ is partly smooth.
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Moreover, if $\bar{x} \in \text{dom } f$ and $\bar{y} \in \partial h(\bar{c})$ are such that $\bar{c} = c(\bar{x})$ and

$$\ker \nabla c(\bar{x})^\top \cap \text{ri } (\partial h(\bar{c})) = \{\bar{y}\}, \quad \text{(Strict Criticality (SC))}$$

then

$$D(\bar{x}) = \{d \mid h'(c(\bar{x})); \nabla c(\bar{x})d \leq 0\} = \ker A^\top \nabla c(\bar{x}).$$
Newton’s Method Hypotheses

Let \( f = h \circ c \) be PLQ convex composite, \( \bar{x} \in \text{dom } f \), \( \bar{y} \in \partial h(c(\bar{x})) \), and set \( \bar{c} := c(\bar{x}) \).

**Assumptions:**

(a) \( c \) is \( \mathcal{C}^3 \)-smooth,

(b) \( M_{\bar{c}} \) satisfies the nondegeneracy condition,

(c) \( f \) satisfies SC at \( \bar{x} \) for \( \bar{y} \),

(d) \( \bar{x} \) satisfies the second-order sufficient conditions, i.e.,
\[
h''(c(\bar{x}); \nabla c(\bar{x})d) + \left\langle d, \nabla^2_{xx} L(\bar{x}, \bar{y})d \right\rangle > 0 \quad \forall \, d \in \ker A^\top \nabla c(\bar{x}) \setminus \{0\},
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where \( M(\bar{x}) = \{\bar{y}\} \) and \( D(\bar{x}) = \ker A^\top \nabla c(\bar{x}) \).
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**NLP Analogues:**

(b) linear independence of the active constraint gradients,

(c) strict complementary slackness, and

(d) strong second-order sufficiency condition.
Convergence of Newton’s Method

There exists a neighborhood $N$ of $(\bar{x}, \bar{y})$ such that if $(x^0, y^0) \in N$, then there exists a unique sequence $\{(x^k, y^k)\}$ satisfying the optimality conditions of $P_k$ with $H_k := \nabla^2_{xx} L(x^k, y^k)$ such that, for all $k \in \mathbb{N}$,

(i) \[ c(x^{k-1}) + \nabla c(x^{k-1})[x^k - x^{k-1}] \in \mathcal{M}_c, \]

(ii) \[ y^k \in \text{ri } \partial h(c(x^{k-1}) + \nabla c(x^{k-1})[x^k - x^{k-1}]), \]

(iii) \[ H_{k-1}[x^k - x^{k-1}] + \nabla c(x^{k-1})^\top y^k = 0, \]

(iv) $x^{k+1}$ is a strong local minimizer of $P_k$.

Moreover, the sequence $(x^k, y^k)$ converges to $(\bar{x}, \bar{y})$ at a quadratic rate.