

Foundations of Gauge and Perspective Duality

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Joint work with
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Dedicated to Prof. B. S. Mordukhovich on the occasion of his 70th birthday
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Thank you Mau Nam Nguyen, Hung M. Phan, and Xianfu (Shawn) Wang!!

Gauge Optimization and Duality

Suppose κ and ρ are gauges.

$$\min_x \quad \kappa(x) \quad \text{s.t.} \quad \rho(b - Ax) \leq \tau, \quad (\text{G}_p)$$

$$\max_y \quad \langle b, y \rangle - \tau \rho^\circ(y) \quad \text{s.t.} \quad \kappa^\circ(A^T y) \leq 1, \quad (\text{L}_d)$$

$$\min_y \quad \kappa^\circ(A^T y) \quad \text{s.t.} \quad \langle b, y \rangle - \tau \rho^\circ(y) \geq 1. \quad (\text{G}_d)$$

When $\tau = 0$, we define $\tau \rho^\circ := \delta_{\text{cl dom } \rho^\circ}$.

Minkowski (gauge) functionals and polarity

Let $0 \in C \subset \mathbb{R}^n$ be nonempty, closed, and convex.

The *gauge function* for C is given by

$$\gamma_C(x) := \inf \{t \mid 0 \leq t, x \in tC\},$$

where the infimum over the empty set is $+\infty$.

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Gauge functions are sublinear, and so by Hörmander,

$$\gamma_C(x) = \sigma_D(x) := \sup \{\langle x, y \rangle \mid y \in D\},$$

where

$$D = \{z \mid \langle z, x \rangle \leq 1 \forall x \in C\} =: C^\circ$$

and σ_D is the *support function* for the set D .

Polar Gauges

Set $\mathcal{U}_\kappa := \{x \mid \kappa(x) \leq 1\}$ and define the *polar gauge* by

$$\kappa^\circ(y) = \sup \{ \langle y, x \rangle \mid \kappa(x) \leq 1 \} = \sigma_{\mathcal{U}_\kappa}(y).$$

If κ is a norm then κ° is the corresponding dual norm.

$$\text{epi } \kappa^\circ = \{(y, -\lambda) : (y, \lambda) \in (\text{epi } \kappa)^\circ\}.$$

The generalized Hölder inequality

$$\langle x, y \rangle \leq \kappa(x) \cdot \kappa^\circ(y) \quad \forall x \in \text{dom } \kappa, \quad \forall y \in \text{dom } \kappa^\circ,$$

is known as the *polar-gauge inequality*.

In addition, for $\mathcal{H}_\kappa := \{u \mid \kappa(u) = 0\}$, we have

$$\mathcal{U}_\kappa^\circ = \mathcal{U}_{\kappa^\circ}, \quad \mathcal{U}_\kappa^\infty = \mathcal{H}_\kappa, \quad (\text{dom } \kappa)^\circ = \mathcal{H}_{\kappa^\circ}, \quad \text{and} \quad \mathcal{H}_\kappa^\circ = \text{cl } \text{dom } \kappa^\circ.$$

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When $\tau = 0$, we define $\tau \rho^\circ := \delta_{\text{cl dom } \rho^\circ}$.

Feasibility

Primal, Dual Domains:

$$\mathcal{F}_p := \{ x \mid \rho(b - Ax) \leq \tau \} \quad \text{and} \quad \mathcal{F}_d := \{ y \mid \langle b, y \rangle - \tau \rho^\circ(y) \geq 1 \}.$$

$$\left. \begin{array}{l} \text{Feasibility :} \\ \text{Primal } \mathcal{F}_p \cap (\text{dom } \kappa) \\ \text{Dual } A^T \mathcal{F}_d \cap (\text{dom } \kappa^\circ) \\ \\ \text{Relative Strict Feasibility :} \\ \text{Primal } \text{ri } \mathcal{F}_p \cap (\text{ri dom } \kappa) \\ \text{Dual } A^T \text{ri } \mathcal{F}_d \cap (\text{ri dom } \kappa^\circ) \\ \\ \text{Strict Feasibility :} \\ \text{Primal } \text{int } (\mathcal{F})_p \cap (\text{ri dom } \kappa) \\ \text{Dual } A^T \text{int } (\mathcal{F})_d \cap (\text{ri dom } \kappa^\circ) \end{array} \right\} \neq \emptyset$$

$$v_p = \min_{\rho(b-Ax) \leq \tau} \kappa(x) \qquad v_d = \min_{\langle b, y \rangle - \tau \rho^\circ(y) \geq 1} \kappa^\circ(A^T y)$$

Theorem: (2014)

1. (Weak duality)

If x and y are P-D feasible, then

$$1 \leq v_p v_d \leq \kappa(x) \cdot \kappa^\circ(A^T y).$$

2. (Strong duality)

If the dual (resp. primal) is feasible and the primal (resp. dual) is relatively strictly feasible, then $v_p v_d = 1$ and the gauge dual (resp. primal) attains its optimal value.

Infimal Projection Duality Theory

Let $F: \mathbb{R}^n \times \mathbb{R}^m \rightarrow \overline{\mathbb{R}}$ be closed proper convex, and define the following optimal value functions by inf-projection:

$$p(y) := \inf_x F(x, y) \quad \text{and} \quad q(w) := \inf_z F^*(w, z).$$

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$$p(0) \geq p^{**}(0) = -q(0) \quad \text{always holds}$$

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Similarly, if $0 \in \text{ri}(\text{dom } q)$, then $p(0) = -q(0)$ and the infimum $p(0)$ is attained, if finite.

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2. The set $\text{argmax}_z -F^*(0, z)$ is nonempty and bounded if and only if $0 \in \text{int}(\text{dom } p)$ and $p(0)$ is finite, in which case $\partial p(0) = \text{argmax}_z -F^*(0, z)$.

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4. Optimal solutions are characterized by

$$\left. \begin{array}{l} \bar{x} \in \text{argmin}_x F(x, 0) \\ \bar{y} \in \text{argmax}_z -F^*(0, z) \\ F(\bar{x}, 0) = -F^*(0, \bar{z}) \end{array} \right\} \iff (0, \bar{z}) \in \partial F(\bar{x}, 0) \iff (\bar{x}, 0) \in \partial F^*(0, \bar{z}).$$

Fenchel-Rockafellar Duality

$$F(x, y) = h(Ax + y) + g(x)$$

$$p(0) = \inf_x \{ h(Ax) + g(x) \} \quad \text{and} \quad p^{**}(0) = \sup_z \{ -h^*(z) - g^*(-A^*z) \}$$

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A prototype problem:

$$\begin{array}{ll} \mathcal{P} & \min \|x\|_1 \\ & \text{s.t. } \|Ax - b\|_2 \leq \tau \end{array}$$

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$$g(x) = \|x\|_1 = \delta^*(x \mid \mathbb{B}_\infty) \quad g^*(w) = \delta(w \mid \mathbb{B}_\infty)$$

$$h(y) = \delta(y - b \mid \tau\mathbb{B}_2) \quad h^*(z) = -\langle z, b \rangle + \delta^*(z \mid \tau\mathbb{B}_2) = -\langle z, b \rangle + \tau\|z\|_2$$

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$$\begin{array}{ll} \mathcal{D}_L & \sup \langle b, z \rangle - \tau\|z\|_2 \\ & \text{s.t. } \|A^T z\|_\infty \leq 1. \end{array}$$

Gauge Duality and Sensitivity

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$$= \inf_{\lambda > 0, w} \{ 1/\lambda \mid \rho(\lambda b - Aw + y) \leq \tau\lambda, w \in \mathcal{U}_\kappa \},$$

or

$$\inf_{\lambda > 0, w} \{ -\lambda \mid \rho(\lambda b - Aw + y) \leq \tau\lambda, w \in \mathcal{U}_\kappa \}.$$

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Variational framework:

$$F(w, \lambda, y) := -\lambda + \delta_{(\text{epi } \rho) \times \mathcal{U}_\kappa} \left(W \begin{pmatrix} w \\ \lambda \\ y \end{pmatrix} \right), \quad W := \begin{pmatrix} -A & b & I \\ 0 & \tau & 0 \\ I & 0 & 0 \end{pmatrix}$$

$$F^*(w, \lambda, y) = \delta_{\text{epi } \rho^\circ} \left(-\sigma^{-1}(1 + \lambda - \langle b, y \rangle) \right) + \kappa^\circ(w + A^T y)$$

Gauge Duality and Sensitivity

$$p(y) := \inf_{w, \lambda} F(w, \lambda, y)$$

Theorem: The following relationships hold for the gauge primal-dual pair G_p and G_d .

- (a) If the primal is relatively strictly feasible and the dual is feasible, then the set of optimal solutions for the dual is nonempty and coincides with

$$\partial p(0) = \partial(-1/v_p)(0).$$

If it is further assumed that the primal is strictly feasible, then the set of optimal solutions to the dual is bounded.

- (b) If the dual is relatively strictly feasible and the primal is feasible, then the set of optimal solutions for the primal is nonempty with solutions $x^* = w^*/\lambda^*$, where

$$(w^*, \lambda^*) \in \partial v_d(0, 0) \text{ and } \lambda^* > 0.$$

If it is further assumed that the dual is strictly feasible, then the set of optimal solutions to the primal is bounded.

Gauge Duality and Optimality Conditions

Theorem: Suppose both the gauge primal and gauge dual problems are relatively strictly feasible, and the pair (x^*, y^*) is primal-dual feasible. Then (x^*, y^*) is primal-dual optimal if and only if it satisfies the conditions

$$\rho(b - Ax^*) = \tau \quad \text{or} \quad \rho^\circ(y^*) = 0 \quad (\text{primal activity})$$

$$\langle b, y^* \rangle - \tau \rho^\circ(y^*) = 1 \quad (\text{dual activity})$$

$$\langle x^*, A^T y^* \rangle = \kappa(x^*) \cdot \kappa^\circ(A^T y^*) \quad (\text{objective alignment})$$

$$\langle bAx^*, y^* \rangle = \tau \rho^\circ(y^*). \quad (\text{constraint alignment})$$

By convention, when $\tau = 0$, $\tau \rho^\circ := \delta_{\text{cl dom } \rho^\circ}$.

Gauge primal-dual recovery

Corollary: Suppose that the primal-dual pair (G_p) and (G_d) are each relatively strictly feasible. If y^* is optimal for (G_d) , then for any primal feasible x the following conditions are equivalent:

- (a) x is optimal for (G_p) ;
- (b) $\langle x, A^T y^* \rangle = \kappa(x) \cdot \kappa^\circ(A^T y^*)$ and $b - Ax \in \partial(\sigma\rho^\circ)(y^*)$;
- (c) $A^T y^* \in \kappa^\circ(A^T y^*) \cdot \partial\kappa(x)$ and $b - Ax \in \partial(\sigma\rho^\circ)(y^*)$,

where, by convention, $\sigma\rho^\circ = \delta_{\text{cl dom } \rho^\circ}$ when $\sigma = 0$, in which case

$$\partial(\sigma\rho^\circ)(y^*) = N(y^* | \mathcal{H}_\rho^\circ).$$

Gauge primal-dual recovery from the Lagrange dual

Theorem:

Suppose that the gauge dual G_d is relatively strictly feasible and the primal G_p is feasible. Let L_p denote the Fenchel-Rockafellar dual of G_d , and let ν_L denote its optimal value. Then

$$z^* \text{ is optimal for } L_p \iff z^*/\nu_L \text{ is optimal for } G_p.$$

Perspective Duality

The Perspective Transform

$$f^\pi(x, \mu) := \begin{cases} \mu f(\mu^{-1}x), & \mu > 0 \\ f^\infty(x), & \mu = 0 \\ +\infty, & \mu < 0 \end{cases}$$

where

$$f^\infty(x) := \sup_{z \in \text{dom}(f)} [f(x+z) - f(x)]$$

is the **horizon** function of f .

$$h^\pi(y, \mu) = \sigma_{\text{epi } h^*}((y, -\mu))$$

The Perspective-Polar Transform

$$\begin{aligned} f^\sharp(x, \xi) &:= (f^\pi)^\circ(x, \xi) \\ &= \sigma_{\text{epi}(f^*)}^\circ(x, -\xi) \\ &= \gamma_{\text{epi}(f^*)}(x, -\xi) \\ &= \inf \{ \mu > 0 \mid \xi + \langle z, x \rangle \leq \mu f(z), \forall z \} \end{aligned}$$

f^\sharp is a gauge.

If f is a gauge, then $f^\sharp(x, \xi) = f^\circ(x) + \delta_{\mathbb{R}_-}(\xi)$.

Perspective duality

Suppose $f : \mathbb{R}^n \rightarrow \bar{\mathbb{R}}_+$ and $g : \mathbb{R}^m \rightarrow \bar{\mathbb{R}}_+$ are closed, convex and nonnegative over their domains.

$$N_p \quad \min_x \quad f(x) \quad \text{s.t.} \quad g(b - Ax) \leq \sigma,$$

$$N_d \quad \min_{y, \alpha, \mu} \quad f^\#(A^T y, \alpha) \quad \text{s.t.} \quad \langle b, y \rangle - \sigma \cdot g^\#(y, \mu) \geq 1 - (\alpha + \mu)$$

The Perspective-Polar of a PLQ Penalty

Piecewise linear-quadratic (PLQ) penalties:

$$g(y) := \sup_{u \in U} \left\{ \langle u, y \rangle - \frac{1}{2} \|Lu\|_2^2 \right\}, \quad U := \left\{ u \in \mathbb{R}^l \mid Wu \leq w \right\},$$

$$\begin{aligned} g^\sharp(y, \mu) &= \delta_{\mathbb{R}_-}(\mu) + \max \left\{ \gamma_U(y), -(1/2\mu) \|Ly\|^2 \right\} \\ &= \delta_{\mathbb{R}_-}(\mu) + \max \left\{ -(1/2\mu) \|Ly\|^2, \max_{i=1, \dots, k} \{ W_i^T y / w_i \} \right\}, \end{aligned}$$

where W_1^T, \dots, W_k^T are the rows of W .

The Perspective Duality for PLQ Penalties

Assume f is a gauge and g is a PLQ penalty, then

$$\min_{(y, \mu, \xi)} f^\circ(A^T y)$$

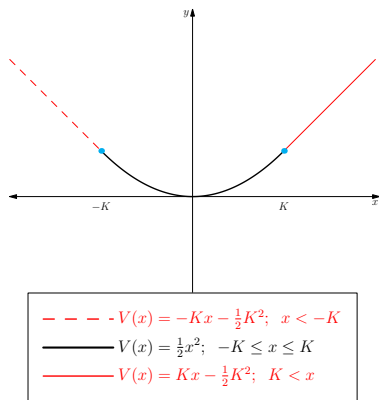
$$\text{s.t.} \quad \langle b, y \rangle + \mu - \sigma\xi = 1$$

$$Wy \leq \xi w, \quad \left\| \begin{bmatrix} 2Ly \\ \xi + 2\mu \end{bmatrix} \right\|_2 \leq \xi - 2\mu$$

Perspective Duality Numerics

$$\begin{aligned} \min_x \quad & \|x\|_1 \\ \text{s.t.} \quad & \sum_{i=1}^m V((Ax - b)_i) \leq \sigma, \end{aligned}$$

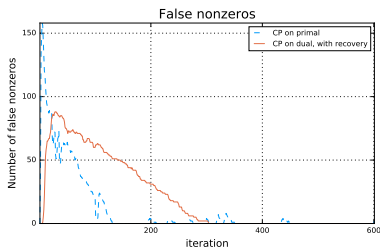
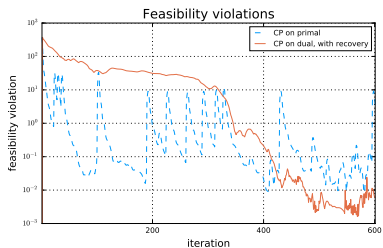
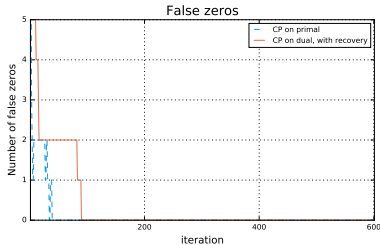
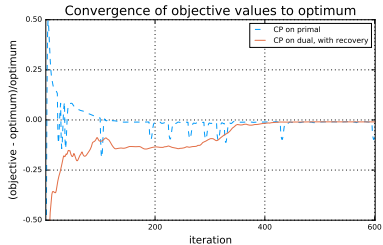
where V is the Huber function



Experiment:

$m = 120$, $n = 512$, $\sigma = 0.2$, $\eta = 1$, and A is a Gaussian matrix. The true solution $x_{\text{true}} \in \{-1, 0, 1\}$ is a spike train which has been constructed to have 20 nonzero entries, and the true noise $b - Ax_{\text{true}}$ has been constructed to have 5 outliers.

Perspective Duality Numerics



Chambolle- Pock (CP) algorithm

THANK YOU BORIS!!

Boris, thank you for your many mathematical gifts over the years!

and thank you for your warm friendship!