1.6 Tangent Cones: First-Order Approximation of Sets

Just as a first-order approximation to function is extremely useful in applications, so are "first-order" approximations of sets. This approximation to a set $S \subset \mathbf{E}$ at a point $x \in S$ is called the *tangent cone* to S at x. Our notion of tangency is based on the *distance to a set* given by

$$\text{dist}_S(y) := \inf_{x \in S} \|y - x\|.$$

Definition 1.17 (Tangent Cone)**.** Let $S \subset \mathbf{E}$. We say that the vector v is tangent to S at a point $\bar{x} \in S$ if for $t > 0$

$$\text{dist}_S(\bar{x} + tv) \leq o(t).$$

We call the set of all such tangent vectors the tangent cone to S at $\bar{x}$ and denote it by $T(\bar{x} \,|\, S)$.

Exercise 1.18. Show that $T(\bar{x} \,|\, S)$ is a closed cone, where a set $K \subset \mathbf{E}$ is said to be a *cone* if $\lambda K \subset K$ for all $\lambda > 0$.

Exercise 1.19. Show that

$$\begin{aligned} T(\bar{x} \,|\, S) &= \left\{ v \,\middle|\, \exists \{x^k\} \subset S,\, t_k \downarrow 0, \text{ s.t. } t_k^{-1}(x^k - \bar{x}) \to v \right\} \\ &= \left\{ tu \,\middle|\, t > 0,\ \exists \{x^k\} \subset S \setminus \{\bar{x}\},\, x^k \to \bar{x},\ \text{s.t. } \frac{x^k - \bar{x}}{\|x^k - \bar{x}\|} \to u \right\} \cup \{0\}. \end{aligned}$$

Exercise 1.20. If C is a nonempty convex subset of $\mathbf{E}$, show that

$$T(\bar{x} \,|\, C) = \text{cl}\,\{t(x - \bar{x}) \,|\, x \in C,\ t \geq 0\}.$$

Exercise 1.21. If C is a convex polyhedron, show that

$$T(\bar{x} \,|\, C) = \{t(x - \bar{x}) \,|\, x \in C,\ t \geq 0\}.$$

Recall that C is a convex polyhedon if there exist $a^i \in E$ and $\beta_i \in \mathbf{R}\, i = 1, \ldots, k$ such that $C = \left\{x \,\middle|\, \left\langle a^i, x \right\rangle \leq \beta_i,\ i = 1, \ldots, k\right\}$.

Exercise 1.22. Let $f : \mathbf{E} \to \mathbf{R}$ be C^1-smooth and set

$$\text{gph} f := \{(x, f(x)) \,|\, x \in \mathbf{E}\}$$

be the *graph* of f. Show that

$$T((\bar{x}, f(\bar{x})) \,|\, \text{gph} f) = \left\{(v, \nabla f(\bar{x})^T v) \,|\, v \in \mathbf{E}\right\}.$$

That is, $T((\bar{x}, f(\bar{x})) \,|\, \text{gph} f)$ is the subspace parallel to the graph of the linearization of f at $\bar{x}$.

The great challenge in using tangent cones is the development of a calculus that is as rich as the one available for differentiable functions.