Convexity and Duality

April 30, 2018

Dual pairs of problems

A prototype problem: $A \in \mathbb{R}^{m \times n}, \; b \in \mathbb{R}^m$

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The (Fenchel-Rockafellar) dual problem:

$$\mathcal{D}_{\mathrm{L}} \qquad \begin{array}{c} \sup \ \langle b, z\rangle - \tau\|z\|_2 \\ \text{s.t. } \left\|A^T z\right\|_\infty \leq 1. \end{array}$$

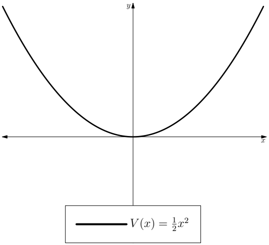
Piecewise Linear-Quadratic Penalties

$$\phi(x) := \sup_{u \in U} [\langle x, u \rangle - \frac{1}{2} u^T B u]$$

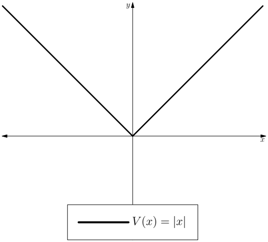
$U \subset \mathbb{R}^n$ is nonempty, closed and convex with $0 \in U$.
$B \in \mathbb{R}^{n \times n}$ is symmetric positive semi-definite.

Examples:
Norms, gauges, support functions, least-squares, Huber density

PLQ Densities: Gauss, Laplace, Huber, Vapnik



Gauss



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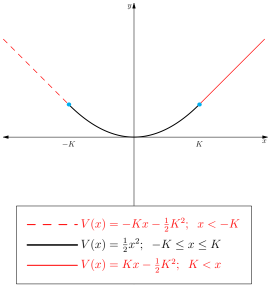
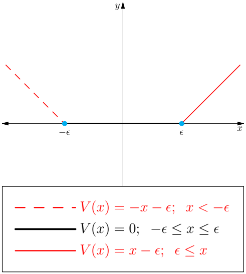


Huber



Vapnik

Convex Sets

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A subset C of $\mathbb{R}^n$ is convex if

$$[x, y] \subset C \quad \forall\, x, y \in C,$$

where

$$[x, y] := \{(1 - \lambda x) + \lambda y \,|\, 0 \leq \lambda \leq 1\}$$

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Convex Cones

A subset K of $\mathbb{R}^n$ is convex if

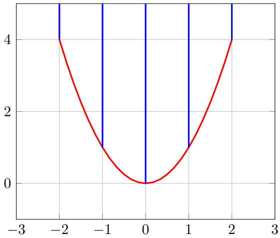
$$\lambda K \subset K \;\; \forall\, \lambda > 0 \;\; \text{and} \; K + K \subset K.$$

Convex functions and the epigraphical perspective

A function $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ is said to be convex if

$$\operatorname{epi} f := \{(x, \mu) \mid f(x) \leq \mu\},$$

is convex.



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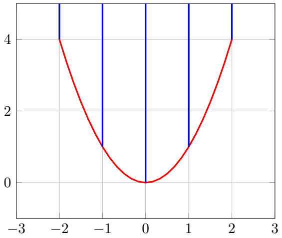

f is lower semi-continuous (lsc) $\iff$ $\operatorname{epi}(f)$ is closed

Linear transformations and their pre-images

If $A, B^T \in \mathbb{R}^{m\times n}$, then both

$$AC := \{Ax \mid x \in C\} \subset \mathbb{R}^m \text{ and}$$
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Convex hull: The convex hull of $S \subset \mathbb{R}^n$ is the intersection of all convex sets in $\mathbb{R}^n$ containing S, denoted $\mathrm{conv}\,(S)$.

Affine sets and relative interior

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Relative interior: The relative interior of a convex set is the interior relative to its affine hull:

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Properties: Let $C \subset \mathbb{R}^n$ be convex and $A, B^T \in \mathbb{R}^{m \times n}$, then

$$A\,\mathrm{ri}\,(C) = \mathrm{ri}\,(AC) \quad \text{and}$$
$$B^{-1}\mathrm{ri}\,(C) = \mathrm{ri}\,(B^{-1}C), \quad \text{whenever } B^{-1}\mathrm{ri}\,(C) \neq \emptyset.$$

The Hahn-Banach Theorem

Hyperplanes: Affine sets of co-dimension 1, or equivalently, any set of the form

$$\{x \mid \langle z, x\rangle = \beta\}$$

for some $\beta \in \mathbb{R}$ and non-zero $z \in \mathcal{L}$.

The Hahn-Banach Theorem: Let C be a non-empty convex set in the Euclidean space $\mathbb{E}$, and let $M \subset \mathbb{E}$ be a nonempty affine set such that

$$M \cap \operatorname{ri} C = \emptyset.$$

Then there is a hyperplane H in $\mathbb{E}$ such that

$$M \subset H \quad \text{and} \quad H \cap \operatorname{ri} C = \emptyset.$$

Supporting hyperplanes

If $\overline{x} \in \operatorname{rbdry}(C) := \operatorname{cl} C \setminus \operatorname{ri} C$, then there is a hyperplane H containing $\overline{x}$ that does not meet the relative interior of C, or equivalently,

$$\exists\, z \text{ s.t. } \langle z, x \rangle < \langle z, \overline{x} \rangle \quad \forall\, x \in \operatorname{ri} C. \tag{1}$$

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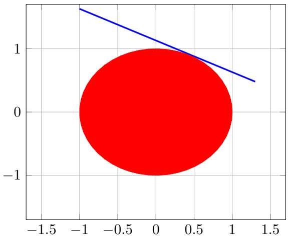

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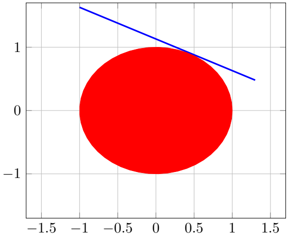

Question: Given $z \in \mathbb{R}^n$, does it define a supporting hyperplane to C and what are the associated support points.

Support functions

The support function for a set $S \subset \mathbb{R}^n$ is given by

$$\sigma_S(z) := \sup_{x \in S} \langle z, x \rangle.$$

It is straightforward to show that

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Hörmander's Theorem: $\sigma : \mathbb{R}^n \to \bar{\mathbb{R}}_+ := \mathbb{R}_+ \cup \{+\infty\}$ lsc.

σ is sublinear $\iff$ epi (σ) is a closed cvx cone $\iff$ $\sigma = \sigma_C$,

where $C := \{z \mid \langle z, x \rangle \leq \sigma(x) \; \forall \, x\} = \{z \mid \langle z, x \rangle \leq 1 \; \forall \, \sigma(x) \leq 1\}$.

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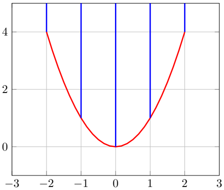

$\operatorname{epi}(f)$

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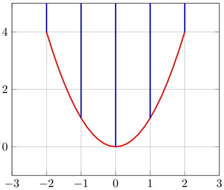
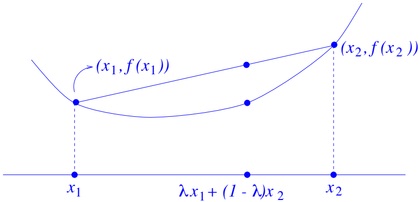


epi (f)



$$f((1-\lambda)x_1 + \lambda x_2) \le (1-\lambda)f(x_1) + \lambda f(x_2)$$

$$\forall\ x_1, x_2 \in \operatorname{dom} f \text{ and } \lambda \in [0, 1]$$

$$\operatorname{dom} f := \{x \mid f(x) < \infty\}$$

Coordinate inf-projection of a convex set

Let $C \subset \mathbb{R}^{m+1}$ be a convex set such that the projection of C onto its last coordinate is bounded below. Define $f : \mathbb{R}^m \to \bar{\mathbb{R}}\}$ by

$$f(x) := \inf \{\overline{x}_{m+1} \mid \exists\, \overline{x} \in C \text{ s.t. } \overline{x} = (x, \overline{x}_{m+1})\},$$

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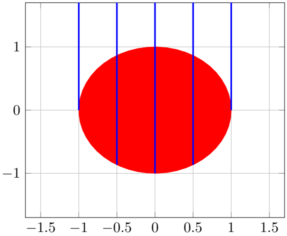

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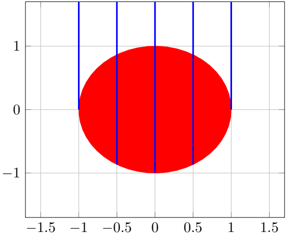

Example: $f(x) := \inf_{(x,\mu) \in \operatorname{epi}(f)} \mu$

Inf-projection: $h(x) := \inf_y f(y, x)$

Let $f : \mathbb{R}^m \times \mathbb{R}^n \to \bar{\mathbb{R}}$ be convex and consider the projection

$$P(y, x, \mu) = (x, \mu).$$

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The support function for epi (f): the convex conjugate

$$\begin{aligned}
\sigma_{\operatorname{epi} f}((z,-1)) &= \sup_{f(x)\le\mu} \langle (z,-1),(x,\mu)\rangle \\
&= \sup_{f(x)\le\mu} [\langle z,x\rangle - \mu] \\
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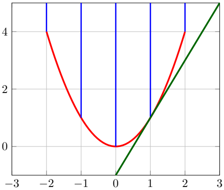
$$\iff f(x) \ge f(\overline{x}) + \langle z, x-\overline{x}\rangle \quad \forall\, x \in \mathbb{R}^n$$

$$z \in \partial f(\overline{x}), \quad \text{the subdifferential of } f \text{ at } \overline{x}.$$

The conjugate and subgradients



$$\partial f(\overline{x}) \text{ is a singleton} \iff \partial f(\overline{x}) = \{\nabla f(\overline{x})\}.$$

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$$f^*(z) \geq \langle z, x \rangle - f(x) \quad \forall \ \ x \in \operatorname{dom}(f) \text{ and } z \in \mathbb{R}^n$$

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So $f(x) = f^{**}(x)$ on $\operatorname{dom}(\partial f)$, where $\operatorname{ri}\operatorname{dom}(f) \subset \operatorname{dom}(\partial f)$.
Consequently $f^{**} = \operatorname{cl} f$, so if $f = \operatorname{cl} f$, $\partial f^* = (\partial f)^{-1}$.

The convex indicator function

$C \subset \mathbb{R}^n$ non-empty closed convex

$$\delta_C(x) := \begin{cases} 0 & , \ x \in C, \\ +\infty & , \ x \notin C \end{cases}$$

$$\delta_C^*(z) = \sigma_C(z)$$

$$\begin{aligned} \partial\delta_C(x) &= \{ z \mid \langle z, y - x \rangle \le 0 \ \ \forall\, y \in C \} \quad (x \in C) \\ &=: N(x \,|\, C) \qquad \text{the normal cone to } C \text{ at } x \\ &= \text{set of supporting vectors to } C \text{ at } x \end{aligned}$$

The conjugate under inf-projection

Let $F : \mathbb{R}^n \times \mathbb{R}^m \to \bar{\mathbb{R}}$ and define the following optimal value function by inf-projection:

$$p(y) := \inf_x F(x, y).$$

Then

$$\begin{aligned} p^*(z) &= \sup_y [\langle z, y \rangle - p(y)] \\ &= \sup_y [\langle z, y \rangle - \inf_x F(x, y)] \\ &= \sup_y \sup_x [\langle z, y \rangle - F(x, y)] \\ &= \sup_{(x,y)} [\langle (0, z), (x, y) \rangle - F(x, y)] \\ &= F^*(0, z) \end{aligned}$$

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Let $F\colon \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}$ and define the following optimal value functions by inf-projection:

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This set-up yields the primal-dual pair

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3. Optimal solutions are characterized by

$$\left.\begin{array}{l} \bar{x} \in \operatorname{argmin}_x \; F(x,0) \\ \bar{y} \in \operatorname{argmax}_z \; -F^*(0,z) \\ F(\bar{x},0) = -F^*(0,\bar{z}) \end{array}\right\} \iff (0,\bar{z}) \in \partial F(\bar{x},0) \iff (\bar{x},0) \in \partial F^*(0,\bar{z}).$$

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The Lagrangian function:

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Fenchel-Rockafellar Duality: $F(x, y) = h(Ax + y) + g(x)$

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$$\mathcal{D}_{\mathrm{L}} \qquad \begin{array}{l} \sup \ \langle b, z\rangle - \tau\|z\|_2 \\ \text{s.t. } \left\|A^T z\right\|_\infty \le 1. \end{array}$$