MATH/AMATH 516
THIRD HOMEWORK SET

(1) Let $\Omega \subset \mathbb{R}^n$ and $x \in \Omega$. The tangent cone to Ω at x is given by

$$T_\Omega(x) := \left\{ u \;\middle|\; \exists \{x^k\} \subset \Omega,\ x^k \to x,\ t_k \downarrow 0 \text{ with } t_k^{-1}(x^k - x) \to u \right\}.$$

(a) Show that

$$T_\Omega(x) := \left\{ tu \;\middle|\; t \ge 0 \text{ and } \exists \{x^k\} \subset \Omega,\ x^k \to x,\ \text{with } \frac{(x^k - x)}{\|x^k - x\|} \to u \right\}.$$

(b) Show that $T_\Omega(x)$ is a cone, i.e. $\lambda T_\Omega(x) \subset T_\Omega(x)$ for all $\lambda \ge 0$.

(2) Let $A \in \mathbb{R}^{s\times n}$, $a \in \mathbb{R}^s$, $B \in \mathbb{R}^{r\times n}$, and $b \in \mathbb{R}^r$. Consider the set $\Omega := \{x \mid Ax \le a,\ Bx = b\}$, where we write $u \le v$ for $u, v \in \mathbb{R}^m$ if and only if $v - u \in \mathbb{R}^m_+ := \{y \in \mathbb{R}^m \mid y_i \ge 0,\ i = 1, \ldots, m\}$, or equivalently, $u_i \le v_i,\ i = 1, \ldots, m$. The set Ω is a convex polyhedron.

(a) Show that Ω is a closed convex set.

(b) Let $\bar{x} \in \Omega$, show that

$$T_\Omega(\bar{x}) = \{v \mid \langle A_{i\cdot}, v\rangle \le 0 \ \forall\, i \in \mathcal{A}(\bar{x}),\ Bv = 0\},$$

where $\mathcal{A}(\bar{x}) := \{i \in \{1, \ldots, s\} \mid \langle A_{i\cdot}, \bar{x}\rangle = a_i\}$ and $A_{i\cdot}$ is the i^{th} row of the matrix A, $i = 1, \ldots, s$.

(c) If $a = 0$ and $b = 0$, show that Ω is a closed convex cone.

(d) Show that the closed set $K \subset \mathbb{E}$ is a convex cone if and only if $K + K \subset K$ and $tK \subset K$ for all $t \ge 0$.

(3) Let $f_i : \mathbb{R}^n \mapsto \mathbb{R}$ be C^1-smooth for $i = 1, \ldots, m$, and let s be an integer such that $1 < s < m$. consider the set

$$\Omega := \{x \in \mathbb{R}^n \mid f_i(x) \le 0,\ i = 1, \ldots, s,\ f_i(x) = 0,\ i = s+1, \ldots, m\}.$$

(a) Show that, for $x \in \Omega$,

$$(\star) \qquad T_\Omega(x) \subset \{v \in \mathbb{R}^n \mid \langle \nabla f_i(x), v\rangle \le 0,\ i \in \mathcal{A}(x),\ \langle \nabla f_i(x), v\rangle = 0,\ i = s+1, \ldots, m\},$$

where $\mathcal{A}(x) := \{i \in \{1, \ldots, s\} \mid f_i(x) = 0\}$.

(b) Suppose $n = 2$ and $s = 2 = m$ so that there are no equalities, and let $f_1(x_1, x_2) := x_2 - x_1^3$ and $f_2(x_2, x_2) := -(x_2 + x_1^3)$. First graph the region Ω and then show that the inclusion in $(\star)$ is strict.

(4) Let $f : \mathbb{R}^n \mapsto \mathbb{R}$ be smooth. We say that f is p-coercive for $p \ge 0$ if

$$\lim_{\|x\|\to\infty} \frac{f(x)}{\|x\|^p} = +\infty.$$

Note that if f is p_0-coercive, then f is p-coercive for all $0 \le p \le p_0$. Define the α-lower level set of f to be the set

$$\text{lev}_f(\alpha) := \{x \in \mathbb{R}^n \mid f(x) \le \alpha\}.$$

Given $Q \in \mathbb{S}^n$ and $b \in \mathbb{R}^n$, set $h(x) := \frac{1}{2}x^TQx - \langle b, x\rangle$.

(a) If $f : \mathbb{R}^n \mapsto \mathbb{R}$ is p-coercive for any $p \ge 0$, show that $\text{lev}_f(\alpha)$ is a compact for all $\alpha \in \mathbb{R}$.

(b) If f is p-coercive and continuous on a nonempty closed set $\Omega \subset \mathbb{R}^n$, show that there must exist a solution to the problem $\min_\Omega f$.

(c) Show that

$$((1-\lambda)x + \lambda y)^TQ((1-\lambda)x + \lambda y) + \lambda(1-\lambda)(x-y)^TQ(x-y) = (1-\lambda)x^TQx + \lambda y^TQy$$

ofr all $x, y \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$.

(d) For what values of p and under what conditions, if any, is the quadratic function h a p-coercive function?

(e) Under what conditions does there exist a solution to $\min_{x\in\mathbb{R}^n} h(x)$?

(f) If there does not exist a solution to $\min_{x\in\mathbb{R}^n} h(x)$, show that the optimal value must be $-\infty$.

(5) Let Ω be a nonempty closed convex subset of $\mathbb{R}^n$, and let $K \subset \mathbb{R}^n$ be a non-empty, closed, convex, cone.

(a) Show that the projection onto Ω, P_Ω, is 1-Lipschitz.

(b) Show that $\phi(x) := \text{dist}\,(x \mid K)$ is positively homogeneous and subadditive, i.e.,

$$\alpha \text{dist}\,(x \mid K) = \text{dist}\,(\alpha x \mid K) \quad \text{and} \quad \text{dist}\,(x + y \mid K) \le \text{dist}\,(x \mid K) + \text{dist}\,(y \mid K)$$

for all $\alpha \ge 0$ and $x, y \in \mathbb{R}^n$.

(c) Show that P_K is positively homogeneous.