Quadratic Programming

1. Introduction

The linear programming model is a very powerful tool for the analysis of a wide variety of problems in the sciences, industry, engineering, and business. However, it does have its limits. Not all phenomena are linear, and once nonlinearities enter the picture an LP model is at best only a first-order approximation. The next level of complexity beyond linear programming is *quadratic programming*. This model allows us to include nonlinearities of a quadratic nature into the objective function. As we shall see this will be a useful tool for including the *Markowitz mean-variance* models of uncertainty in the selection of optimal portfolios.

A quadratic program (QP) is an optimization problem wherein one either minimizes or maximizes a quadratic objective function of a finite number of decision variable subject to a finite number of linear inequality and/or equality constraints. A *quadratic* function of a finite number of variables $x = (x_1, x_2, \ldots, x_n)^T$ is any function of the form

$$f(x) = \alpha \ + \ \sum_{j=1}^{n} c_j x_j \ + \ \frac{1}{2} \sum_{k=1}^{n} \sum_{j=1}^{n} q_{kj} x_k x_j .$$

Using matrix notation, this expression simplifies to

$$f(x) \ = \ \alpha \ + \ c^T x \ + \ \frac{1}{2} x^T Q x,$$

where

$$c = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \qquad \text{and} \qquad Q = \begin{pmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} & \cdots & q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ q_{n1} & q_{n2} & \cdots & q_{nn} \end{pmatrix} .$$

The factor of one half preceeding the quadratic term in the function f is included for the sake of convenience since it simplifies the expressions for the first and second derivatives of f. With no loss in generality, we may as well assume that the matrix Q is symmetric since

$$x^T Q x = (x^T Q x)^T = x^T Q^T x = \frac{1}{2}(x^T Q x + x^T Q^T x) = x^T (\frac{Q + Q^T}{2}) x,$$

and so we are free to replace the matric Q by the symmetric matrix $\frac{Q+Q^T}{2}$. Henceforth, we will assume that the matrix Q is symmetric.

The QP *standard form* that we use is

$$\mathcal{Q} \qquad \begin{array}{ll} \text{minimize} & c^T x + \frac{1}{2} x^T Q x \\ \text{subject to} & Ax \le b,\ 0 \le x, \end{array}$$

where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Just as in the case of linear programming, every quadratic program can be transformed into one in standard form. Observed that we have simplified the expression for the objective function by dropping the constant term α since it plays no role in the optimization step.

2. Optimality Conditions

The first step in the analysis of the problem $\mathcal{Q}$ is to derive conditions that allow us to recognize when a particular vector $\bar{x}$ is a solution, or local solution, to the problem. For example, when we minimize a function of one variable we first take the derivative and see if it is zero. If it is, then we take the second derivative and check that it is positive. If this is also true, then we know that the point under consideration is a local minimizer of the function. When constraints are present, the unconstrained optimality employed are the KKT conditions that we derived in our study of constrained optimization. We now describe these conditions in the context of quadratic programming.

The Lagrangian for $\mathcal{Q}$ is the function

$$L(x, y, z) = \frac{1}{2} x^T Q x + c^T x + y^T (Ax - b) - z^T x,$$

where $0 \le y$ and $0 \le z$. The KKT conditions become

- : (i) $Ax \le b$ and $0 \le x$, (*primal feasibility*)
- : (ii) $0 \le y$ and $0 \le z$ (*dual feasibility*)
- : (iii) $y^T (Ax - b) = 0$ and $z^T x = 0$ (*complementarity*)
- : (iv) $0 = \nabla_x L(x, y, z) = Qx + c + A^T y - z$ (*stationarity*)

Note that we can drop the variable z from these conditions by making use of condition (iv) to write $z = Qx + c + A^T y$. The resulting conditions yield what we will call the KKT conditions for $\mathcal{Q}$.

Definition 2.1. *(Karush-Kuhn-Tucker Conditions for $\mathcal{Q}$)*
A pair $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$ is said to be a Karush-Kuhn-Tucker pair (or KKT pair) for the quadratic program $\mathcal{Q}$ if and only if the following conditions are satisfied:

$$\begin{array}{rl} 0 \le x,\ Ax \le b & (\textit{primal feasibility}) \\ 0 \le y,\ 0 \le c + Qx + A^T y, & (\textit{dual feasibility}) \\ 0 = y^T (Ax - b),\ \textit{and}\ 0 = x^T (c + Qx + A^T y)\ . & (\textit{complementarity}) \end{array}$$

Theorem 2.2. *(First-Order Necessary Conditions for Optimality in Quadratic Programming) If $\bar{x}$ solves $\mathcal{Q}$, then there exists a vector $\bar{y} \in \mathbb{R}^m$ such that $(\bar{x}, \bar{y})$ is a KKT pair for $\mathcal{Q}$.*

We now consider the convex case. The constraint region in $\mathcal{Q}$ is already a polyhedral convex set, so we need only consider the objective $f(x) = \frac{1}{2}x^T Qx + c^T x$. From our work with convex functions we know that f is convex if and only if $\nabla^2 f(x) = Q$ is positive semi-definite (assuming Q is symmetric). Our results on KKT conditions in the convex case yield the following result.

Theorem 2.3. *[Necessary and Sufficient Condition for Optimality in Convex Quadratic Programming]*
If Q is symmetric and positive semi-definite, the $\bar{x}$ solves $\mathcal{Q}$ if and only if there exists $\bar{y} \in \mathbb{R}^m$ such that $(\bar{x}, \bar{y})$ is a KKT pair for $\mathcal{Q}$.

Exercises

(1) Write the following quadratic functions in matrix form:
   (a) $f(x_1, x_2) = 5x_2^2 - 2x_2x_2 + 5x_2^2$.
   (b) $f(x_1, x - 2) = 2x_1^2 + x_2^2 - 2x_1x_2 - 5x_1 - 2x_2$.
   (c) $f(x, y, z) = (x-1)^2 - 2(x-1)(y-3) + 2(y-3)^2$.
(2) Determine if the matrices obtained in problem 1 are positive definite, positive semi-definite, or neither.
(3) What are the KKT conditions for the following QPs?
   (a)

$$\begin{array}{ll} \text{minimize} & 2x_1^2 + x_2^2 - 2x_1x_2 - 5x_1 - 2x_2 \\ \text{subject to} & 3x_1 + 2x_2 \leq 20 \\ & -5x_1 + 3x_2 \leq 4 \\ & 0 \leq x_1,\ 0 \leq x_2\ . \end{array}$$

   (b)

$$\begin{array}{ll} \text{minimize} & 5x_2^2 - 2x_1x_2 + 5x_2^2 \\ & x_1 + x_2 = 1 \\ & 0 \leq x_1,\ 0 \leq x_2\ . \end{array}$$

(4) Let $0 \leq \delta$, $Q \in \mathbb{R}^{n\times n}$ be symmetric and positive definite, and $m \in \mathbb{R}^n$ be such that each component of m is positive. Define the vector $\mathrm{e} \in \mathbb{R}^n$ to be the vector each of whose components is the number 1. We will further assume that the vectors m and e are linearly independent. Consider the convex quadratic

program

$$\begin{array}{lll} \mathcal{M} & \text{miminize} & \frac{1}{2}x^TQx \\ & \text{subject to} & \mathrm{e}^Tx = 1,\ m^Tx \geq \delta. \end{array}$$

(a) Show that a solution to this quadratic program must exist. (This is a *bonus* question. You are not required to know how to answer on a quiz or exam.)

(b) Use the KKT conditions for this problem to show that either

$$\frac{m^TQ^{-1}\mathrm{e}}{\mathrm{e}^TQ^{-1}\mathrm{e}} \geq \delta$$

in which case

$$x_{\mathrm{mv}} = \frac{1}{\mathrm{e}^TQ^{-1}\mathrm{e}}Q^{-1}\mathrm{e}$$

solves $\mathcal{M}$, or the solution to $\mathcal{M}$ is given by

$$\bar{x} = \alpha Q^{-1}e + \beta Q^{-1}m\ ,$$

where α and β is the unique solution to the 2×2 system

$$\begin{bmatrix} \mathrm{e}^TQ^{-1}\mathrm{e} & \mathrm{e}^TQ^{-1}m \\ \mathrm{e}^TQ^{-1}m & m^TQ^{-1}m \end{bmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 \\ \delta \end{pmatrix}.$$

(5) Consider the quadratic program

$$\begin{array}{lll} \mathcal{Q}_0 & \text{minimize} & c^Tu + p^Tv + \frac{1}{2}\left[u^TQu + 2v^TMu + v^THv\right] \\ & \text{subject to} & Au + Bv \leq r \\ & & Eu + Fv = h \\ & & 0 \leq u\ , \end{array}$$

where

$$A \in \mathbb{R}^{m\times n},\quad B \in \mathbb{R}^{m\times t},\quad E \in \mathbb{R}^{s\times n},\quad F \in \mathbb{R}^{s\times t},\quad \text{and}\quad M \in \mathbb{R}^{t\times n},$$

and

$$Q \in \mathbb{R}^{n\times n} \quad \text{and} \quad H \in \mathbb{R}^{t\times t} \quad \text{are symmetric},$$

and

$$c \in \mathbb{R}^n,\quad p \in \mathbb{R}^t,\quad r \in \mathbb{R}^m,\quad \text{and}\quad h \in \mathbb{R}^s.$$

Follow the proceedure given above for the quadratic program $\mathcal{Q}$ to show that one can derive the following set of first-order optimality conditions (KKT conditions) for the problem $\mathcal{Q}_0$:

$$
\begin{aligned}
Au + Bv &\le r, \\
Eu + Fv &= h, \\
0 &\le u, \\
0 &\le c + Qu + M^T v + A^T y + E^T w, \\
0 &= p + Hv + Mu + B^T y + F^T w, \\
0 &\le y, \\
0 &= u^T \left[c + Qu + M^T v + A^T y + E^T w \right], \\
0 &= y^T \left[Au + Bv - r \right] \; .
\end{aligned}
$$