OPTIMALITY CONDITIONS

1. Unconstrained Optimization

1.1. **Existence.** Consider the problem of minimizing the function $f : \mathbb{R}^n \to \mathbb{R}$ where f is continuous on all of $\mathbb{R}^n$:

$$\mathcal{P} \qquad \min_{x \in \mathbb{R}^n} f(x).$$

As we have seen there is no guarantee that f has a minimum value, or if it does, there is no guarantee that it is attained. In this regard, the first issue we address is existence of solutions to $\mathcal{P}$. In particular, we are interested in conditions under which we can be certain that a global solution to $\mathcal{P}$ exists.

Recall that we already have at our disposal a rudimentary existence result for constrained problems. This is the Weierstrass extreme value theorem.

Theorem 1.1. *(Weierstrass Extreme Value Theorem) Every continuous function on a compact set attains its extreme values on that set.*

We now build a basic existence result for unconstrained problems on this theorem. For this we make use of the notion of a coercive function.

Definition 1.1. *A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be coercive if for every sequence $\{x^\nu\} \subset \mathbb{R}^n$ for which $\|x^\nu\| \to \infty$ it must be the case that $f(x^\nu) \to \infty$ as well.*

Continuous coercive functions can be characterized by an underlying compactness property on their lower level sets.

Theorem 1.2. *(Coercivity and Compactness) Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous on all of $\mathbb{R}^n$. The function f is coercive if and only if for every $\alpha \in \mathbb{R}$ the set $\{x \mid f(x) \le \alpha\}$ is compact.*

Proof. We first show that the coercivity of f implies the compactness of the sets $\{x \mid f(x) \le \alpha\}$. We begin by noting that the continuity of f implies the closedness of the sets $\{x \mid f(x) \le \alpha\}$. Thus, by the Bolzano-Weierstrass Theorem, it remains only to show that any set of the form $\{x \mid f(x) \le \alpha\}$ is bounded. We show this by contradiction. Suppose to the contrary that there is an $\alpha \in \mathbb{R}^n$ such that the set $S = \{x \mid f(x) \le \alpha\}$ is unbounded. Then there must exist a sequence $\{x^\nu\} \subset S$ with $\|x^\nu\| \to \infty$. But then, by the coercivity of f, we must also have $f(x^\nu) \to \infty$. This contradicts the fact that $f(x^\nu) \le \alpha$ for all $\nu = 1, 2, \ldots$. Therefore the set S must be bounded.

Let us now assume that each of the sets $\{x \mid f(x) \le \alpha\}$ is bounded and let $\{x^\nu\} \subset \mathbb{R}^n$ be such that $\|x^\nu\| \to \infty$. Let us suppose that there exists a subsequence of the integers $J \subset \mathbb{N}$ such that the set $\{f(x^\nu)\}_J$ is bounded above. Then there exists $\alpha \in \mathbb{R}^n$ such that $\{x^\nu\}_J \subset \{x \mid f(x) \le \alpha\}$. But this cannot be the case since each of the sets $\{x \mid f(x) \le \alpha\}$ is bounded while every subsequence of the sequence $\{x^\nu\}$ is unbounded by definition. Therefore, the set $\{f(x^\nu)\}_J$ cannot be bounded, and so the sequence $\{f(x^\nu)\}$ contains no bounded subsequence, i.e. $f(x^\nu) \to \infty$. $\square$

This result in conjunction with Weierstrass's Theorem immediately yields the following existence result for the problem $\mathcal{P}$.

Theorem 1.3. *(Coercivity implies existence) Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous on all of $\mathbb{R}^n$. If f is coercive, then f has at least one global minimizer.*

Proof. Let $\alpha \in \mathbb{R}$ be chosen so that the set $S = \{x \,| f(x) \leq \alpha\}$ is non-empty. By coercivity, this set is compact. By Weierstrass's Theorem, the problem $\min\{f(x) \,| x \in S\}$ has at least one global solution. Obviously, the set of global solutions to the problem $\min\{f(x) \,| x \in S\}$ is a global solution to $\mathcal{P}$ which proves the result. □

1.2. **First-Order Optimality Conditions.** This existence result can be quite useful, but unfortunately it does not give us a constructive test for optimality. That is, we may know a solution exists, but we still do not have a method for determining whether any given point may or may not be a solution. We now build such a test on the derivatives of the objective function f. For this we will assume that f is twice continuously differentiable on $\mathbb{R}^n$ and develop constructible first- and second-order necessary and sufficient conditions for optimality.

The optimality conditions we consider are built up from those developed in first term calculus for functions mapping from $\mathbb{R}$ to $\mathbb{R}$. The reduction to the one dimensional case comes about by considering the functions $\phi : \mathbb{R} \to \mathbb{R}$ given by

$$\phi(t) = f(x + td)$$

for some choice of x and d in $\mathbb{R}^n$. The key variational object in this context is the directional derivative of f at a point x in the direction d given by

$$f'(x; d) = \lim_{t \downarrow 0} \frac{f(x + td) - f(x)}{t}.$$

When f is differentiable at the point $x \in \mathbb{R}^n$, then

$$f'(x; d) = \nabla f(x)^T d = \phi'(0).$$

Note that if $f'(x; d) < 0$, then there must be a $\bar{t} > 0$ such that

$$\frac{f(x + td) - f(x)}{t} < 0 \quad \text{whenever} \quad 0 < t < \bar{t}\,.$$

In this case, we must have

$$f(x + td) < f(x) \quad \text{whenever} \quad 0 < t < \bar{t}\,.$$

That is, we can always reduce the function value at x by moving in the direction d an arbitrarily small amount. In particular, if there is a direction d such that $f'(x; d)$ exists with $f'(x; d) < 0$, then x cannot be a local solution to the problem $\min_{x \in \mathbb{R}^n} f(x)$. Or equivalently, if x is a local to the problem $\min_{x \in \mathbb{R}^n} f(x)$, then $f'(x; d) \geq 0$ whenever $f'(x; d)$ exists. We state this elementary result in the following lemma.

Lemma 1.1 (Basic First-Order Optimality Result)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}$ and let $\bar{x} \in \mathbb{R}^n$ be a local solution to the problem* $\min_{x \in \mathbb{R}^n} f(x)$*. The*

$$f'(x; d) \geq 0$$

for every direction $d \in \mathbb{R}^n$ for which $f'(x; d)$ exists.

We now apply this result to the case in which $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable.

Theorem 1.4. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable at a point $\overline{x} \in \mathbb{R}^n$. If $\overline{x}$ is a local minimum of f, then $\nabla f(\overline{x}) = 0$.*

Proof. By Lemma 1.1 we have

$$0 \le f'(\bar{x}; d) = \nabla f(\bar{x})^T d \quad \text{for all} \quad d \in \mathbb{R}^n \ .$$

Taking $d = -\nabla f(\bar{x})$ we get

$$0 \le -\nabla f(\bar{x})^T \nabla f(\bar{x}) = -\|\nabla f(\bar{x})\|^2 \le 0.$$

Therefore, $\nabla f(\bar{x}) = 0$. □

When $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable, any point $x \in \mathbb{R}^n$ satisfying $\nabla f(x) = 0$ is said to be a stationary (or, equvalently, critical) point of f. In our next result we link the notions of coercivity and stationarity.

Theorem 1.5. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable on all of $\mathbb{R}^n$. If f is coercive, then f has at least one global minimizer these global minimizers can be found from among the critical points of f.*

Proof. Since differentiability implies continuity, we already know that f has at least one global minimizer. Differentiabilty implies that this global minimizer is critical. □

This result indicates that one way to find a global minimizer of a coercive differentiable function is to first find all critical points and then from among these determine those yielding the smallest function value.

1.3. **Second-Order Optimality Conditions.** To obtain second-order conditions for optimality we must first recall a few properties of the Hessian matrix $\nabla^2 f(x)$. The calculus tells us that if f is twice continuously differentiable at a point $x \in \mathbb{R}^n$, then the Hessian $\nabla^2 f(x)$ is a symmetric matrix. Perhaps the most noteworthy feature of symmetric matrices is that they are orthogonally diadonalizable. That is there exists and orthonormal basis of eigenvectors of $\nabla^2 f(x)$, $v^1, v^2, \ldots, v^n$ such that

$$\nabla^2 f(x) = V^T \begin{bmatrix} \lambda_1 & 0 & 0 & \ldots & 0 \\ 0 & \lambda_2 & 0 & \ldots & 0 \\ \vdots & & \ddots & \vdots & \\ 0 & 0 & \ldots & \ldots & \lambda_n \end{bmatrix} V$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $\nabla^2 f(x)$ and V is the matrix whose columns are given by their corresponding vectors $v^1, v^2, \ldots, v^n$:

$$V = \left[v^1, v^2, \ldots, v^n\right] \ .$$

The symmetric matrix $\nabla^2 f(x)$ is said to be positive semi-definite if $\lambda_i \ge 0, \ i = 1, 2, \ldots, n$, and it is said to be positive definite if $\lambda_i > 0, \ i = 1, 2, \ldots, n$. Thus, in particular, if $\nabla^2 f(x)$ is positive definite, then

$$d^T \nabla^2 f(x) d \ge \lambda_{\min} \|d\|^2 \quad \text{for all } d \in \mathbb{R}^n,$$

where $\lambda_{\min}$ is the smallest eigenvalue of $\nabla^2 f(x)$.

We now give our main result on second-order necessary and sufficient conditions for optimality in the problem $\min_{x\in\mathbb{R}^n} f(x)$. The key tools in the proof are the notions of positive semi-definiteness and definiteness along with the second-order Taylor series expansion for f at a given point $\overline{x} \in \mathbb{R}^n$:

$$f(x) = f(\overline{x}) + \nabla f(\overline{x})^T(x-\overline{x}) + \frac{1}{2}(x-\overline{x})^T\nabla^2 f(\overline{x})(x-\overline{x}) + o(\|x-\overline{x}\|^2) \tag{1.1}$$

where $\lim\limits_{x\to\overline{x}} \dfrac{o(\|x-\overline{x}\|^2)}{\|x-\overline{x}\|^2} = 0$.

Theorem 1.6. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be twice continuously differentiable at the point $\overline{x} \in \mathbb{R}^n$.*

(1) *(Necessity) If $\overline{x}$ is a local minimum of f, then $\nabla f(\overline{x}) = 0$ and $\nabla^2 f(\overline{x})$ is positive semi-definite.*

(2) *(Sufficiency) If $\nabla f(\overline{x}) = 0$ and $\nabla^2 f(\overline{x})$ is positive definite, then there is an $\alpha > 0$ such that $f(x) \geq f(\overline{x}) + \alpha\|x-\overline{x}\|^2$ for all x near $\overline{x}$.*

Proof. (1) We make use of the second order Taylor series expansion (1.1) and the fact that $\nabla f(\bar{x}) = 0$ by Theorem 1.4. Given $d \in \mathbb{R}^n$ and $t > 0$ set $x := \overline{x} + td$, plugging this into (1.1) we find that

$$0 \leq \frac{f(x+td) - f(x)}{t^2} = \frac{1}{2}d^T\nabla^2 f(\overline{x})d + \frac{o(t^2)}{t^2}$$

since $\nabla f(\overline{x}) = 0$ by Theorem 1.4. Taking the limit as $t \to 0$ we get that

$$0 \leq d^T\nabla^2 f(\overline{x})d.$$

Now since d was chosen arbitrarily we have that $\nabla^2 f(\overline{x})$ is positive semi-definite.

(2) The Taylor expansion (1.1) and the hypothesis that $\nabla f(\bar{x}) = 0$ imply that

$$\frac{f(x) - f(\overline{x})}{\|x-\overline{x}\|^2} = \frac{1}{2}\frac{(x-\overline{x})^T}{\|x-\overline{x}\|}\nabla^2 f(\overline{x})\frac{(x-\overline{x})}{\|x-\overline{x}\|} + \frac{o(\|x-\overline{x}\|^2)}{\|x-\overline{x}\|^2}. \tag{1.2}$$

If $\lambda_{\min} > 0$ is the smallest eigenvalue of $\nabla^2 f(\overline{x})$, choose $\epsilon > 0$ so that

$$\left|\frac{o(\|x-\overline{x}\|^2)}{\|x-\overline{x}\|^2}\right| \leq \frac{\lambda_{\min}}{4} \tag{1.3}$$

whenever $\|x-\overline{x}\| < \epsilon$. Then for all $\|x-\overline{x}\| < \epsilon$ we have from (1.2) and (1.3) that

$$\begin{aligned} \tfrac{f(x)-f(\overline{x})}{\|x-\overline{x}\|^2} &\geq \tfrac{1}{2}\lambda_{\min} + \tfrac{o(\|x-\overline{x}\|^2)}{\|x-\overline{x}\|^2} \\ &\geq \tfrac{1}{4}\lambda_{\min}. \end{aligned}$$

Consequently, if we set $\alpha = \frac{1}{4}\lambda_{\min}$, then

$$f(x) \geq f(\overline{x}) + \alpha\|x-\overline{x}\|^2$$

whenever $\|x-\overline{x}\| < \epsilon$.

□

1.4. **Convexity.** We have now established first-order necessary conditions and second-order necessary and sufficient conditions. What about first-order sufficiency conditions? For this we introduce the following definitions.

Definition 1.2. (1) *A set $C \subset \mathbb{R}^n$ is said to be convex if for every $x, y \in C$ and $\lambda \in [0,1]$ one has*

$$(1-\lambda)x + \lambda y \in C \ .$$

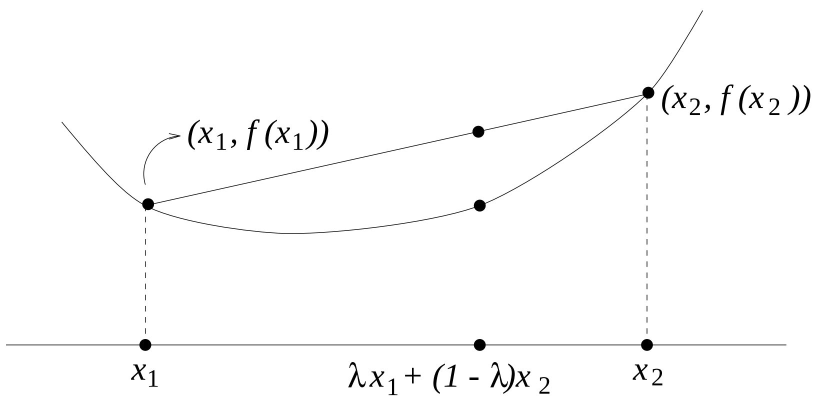
(2) *A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be convex if for every two points $x_1, x_2 \in \mathbb{R}^n$ and $\lambda \in [0,1]$ we have*

$$f(\lambda x_1 + (1-\lambda)x_2) \le \lambda f(x_1) + (1-\lambda) f(x_2).$$

The function f is said to be strictly convex if for every two distinct points $x_1, x_2 \in \mathbb{R}^n$ and $\lambda \in [0,1]$ we have

$$f(\lambda x_1 + (1-\lambda)x_2) < \lambda f(x_1) + (1-\lambda) f(x_2). \tag{1.4}$$

The secant line connecting $(x_1, f(x_1))$ and $(x_2, f(x_2))$ lies above the graph of f.



That is, the set

$$\text{epi}\,(f) = \{(x,\mu) \ : \ f(x) \le \mu\},$$

called the *epi-graph* of f is a convex set. Indeed, it can be shown that the convexity of the set epi (f) is equivalent to the convexity of the function f. This observation allows us to extend the definition of the convexity of a function to functions taking potentially infinite values.

Definition 1.3. *A function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} = \bar{\mathbb{R}}$ is said to be convex if the set epi $(f) = \{(x,\mu) \ : \ f(x) \le \mu\}$ is a convex set. We also define the essential domain of f to be the set*

$$dom\,(f) = \{x \ : \ f(x) < +\infty\} \ .$$

We say that f is strictly convex if the strict inequality (1.4) holds whenever $x_1, x_2 \in dom\,(f)$ are distinct.

Example: $c^T x$, $\|x\|$, e^x, x^2

Convexity is a very important property in optimization theory as is illustrated in the following result.

Theorem 1.7. *Let $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ be convex. If $\overline{x} \in \mathbb{R}^n$ is a local minimum for f, then $\overline{x}$ is a global minimum for f.*

Proof. Suppose there is a $\widehat{x} \in \mathbb{R}^n$ with $f(\widehat{x}) < f(\overline{x})$. Let $\epsilon > 0$ be such that

$$f(\overline{x}) \leq f(x) \quad \text{whenever} \quad \|x - \overline{x}\| \leq \epsilon$$

and

$$\epsilon < 2\|\overline{x} - \widehat{x}\| \ .$$

Set $\lambda := \epsilon(2\|\overline{x} - \widehat{x}\|)^{-1} < 1$ and $x_\lambda := \overline{x} + \lambda(\widehat{x} - \overline{x})$. Then $\|x_\lambda - \overline{x}\| \leq \epsilon/2$ and $f(x_\lambda) \leq (1-\lambda)f(\overline{x}) + \lambda f(\widehat{x}) < f(\overline{x})$. This contradicts the choice of ϵ and so no such $\widehat{x}$ exists. □

Strict convexity implies the uniqueness of solutions.

Theorem 1.8. *Let $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ be strictly convex. If f has a global minimizer, then it is unique.*

Proof. Let x^1 and x^2 be distinct global minimizers of f. Then, for $\lambda \in (0, 1)$,

$$f((1-\lambda)x^1 + \lambda x^2) < (1-\lambda)f(x^1) + \lambda f(x^2) = f(x_1) \ ,$$

which contradicts the assumption that x^1 is a global minimizer. □

If f is a differentiable convex function, much more can be said. We begin with the following lemma.

Lemma 1.2. *Let $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ be convex (not necessarilty differentiable).*

(1) *Given $x, d \in \mathbb{R}^n$ the difference quotient*

$$\frac{f(x+td) - f(x)}{t} \tag{1.5}$$

is a non-decreasing function of t on $(0, +\infty)$.

(2) *For every $x, d \in \mathbb{R}^n$ the directional derivative $f'(x; d)$ always exists and is given by*

$$f'(x; d) := \inf_{t>0} \frac{f(x+td) - f(x)}{t}. \tag{1.6}$$

Proof. We assume (1) is true and show (2). Recall that

$$f'(x; d) := \lim_{t \downarrow 0} \frac{f(x+td) - f(x)}{t}. \tag{1.7}$$

Now if the difference quotient (1.5) is non-decreasing in t on $(0, +\infty)$, then the limit in (1.7) is necessarily given by the infimum in (1.6). This infimum always exists and so $f'(x; d)$ always exists and is given by (1.6).

We now prove (1). Let $x, d \in \mathbb{R}^n$ and let $0 < t_1 < t_2$. Then

$$\begin{aligned} f(x + t_1 d) &= f\left(x + \left(\tfrac{t_1}{t_2}\right) t_2 d\right) \\ &= f\left[\left(1 - \left(\tfrac{t_1}{t_2}\right)\right) x + \left(\tfrac{t_1}{t_2}\right)(x + t_2 d)\right] \\ &\leq \left(1 - \tfrac{t_1}{t_2}\right) f(x) + \left(\tfrac{t_1}{t_2}\right) f(x + t_2 d). \end{aligned}$$

Hence

$$\frac{f(x+t_1d)-f(x)}{t_1} \leq \frac{f(x+t_2d)-f(x)}{t_2}.$$

□

A very important consequence of Lemma 1.2 is the *subdifferential inequality*. This inequality is obtained by plugging $t = 1$ and $d = y - x$ into the right hand side of (1.6) where y is *any* other point in $\mathbb{R}^n$. This substitution gives the inequality

$$(1.8) \qquad f(y) \geq f(x) + f'(x; y-x) \quad \text{for all} \quad y \in \mathbb{R}^n \text{ and } x \in \text{dom}\,(f)\ .$$

The subdifferential inequality immediately yields the following result.

Theorem 1.9 (Convexity and Optimality). *Let $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ be convex (not necessarilty differentiable) and let $\bar{x} \in \text{dom}\,(f)$. Then the following three statements are equivalent.*

(i) *$\bar{x}$ is a local solution to $\min_{x\in\mathbb{R}^n} f(x)$.*
(ii) *$f'(\bar{x}; d) \geq 0$ for all $d \in \mathbb{R}^n$.*
(iii) *$\bar{x}$ is a global solution to $\min_{x\in\mathbb{R}^n} f(x)$.*

Proof. Lemma 1.1 gives the implication (i)⇒(ii). To see the implication (ii)⇒(iii) first note that $f'(\bar{x}; y - \bar{x})$ exists for all $y \in \mathbb{R}^n$ by Lemma 1.2 and then apply the subdifferential inequality (1.8) to obtain

$$f(y) \geq f(\bar{x}) + f'(\bar{x}; y-\bar{x}) \geq f(\bar{x}) \quad \text{for all } y \in \mathbb{R}^n.$$

The implication (iii)⇒(i) is obvious. □

If it is further assumed that f is differentiable, then we obtain the following elementary consequence of Theorem 1.9.

Theorem 1.10. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex and suppose that $\overline{x} \in \mathbb{R}^n$ is a point at which f is differentiable. Then $\overline{x}$ is a global minimum of f if and only if $\nabla f(\overline{x}) = 0$.*

As Theorems 1.9 and 1.10 demonstrate, convex functions are very nice functions to deal with when it comes to optimization theory. Thus it is important that we be able to recognize when a function is convex. For this reason we give the following result.

Theorem 1.11. *Let $f : \mathbb{R}^n \to \bar{\mathbb{R}}$.*

(1) *If f is differentiable on $\mathbb{R}^n$, then the following statements are equivalent:*
  (a) *f is convex,*
  (b) *$f(y) \geq f(x) + \nabla f(x)^T(y - x)$ for all $x, y \in \mathbb{R}^n$*
  (c) *$(\nabla f(x) - \nabla f(y))^T(x - y) \geq 0$ for all $x, y \in \mathbb{R}^n$.*
(2) *If f is twice differentiable then f is convex if and only if f is positive semi-definite for all $x \in \mathbb{R}^n$.*

Proof. (a) ⇒ (b) If f is convex, then 1.11 holds. By setting $t := 1$ and $d := y - x$ we obtain (b).

(b) ⇒ (c) Let $x, y \in \mathbb{R}^n$. From (b) we have

$$f(y) \geq f(x) + \nabla f(x)^T(y - x)$$

and

$$f(x) \geq f(y) + \nabla f(y)^T (x - y).$$

By adding these two inequalities we obtain (c).

(c) $\Rightarrow$ (b)   Let $x, y \in \mathbb{R}^n$. By the mean value theorem there exists $0 < \lambda < 1$ such that

$$f(y) - f(x) = \nabla f(x_\lambda)^T (y - x)$$

where $x_\lambda := \lambda y + (1 - \lambda)x$. By hypothesis,

$$\begin{aligned} 0 &\leq [\nabla f(x_\lambda) - \nabla f(x)]^T (x_\lambda - x) \\ &= \lambda [\nabla f(x_\lambda) - \nabla f(x)]^T (y - x) \\ &= \lambda [f(y) - f(x) - \nabla f(x)^T (y - x)]. \end{aligned}$$

Hence $f(y) \geq f(x) + \nabla f(x)^T (y - x)$.

(b) $\Rightarrow$ (a)   Let $x, y \in \mathbb{R}^n$ and set

$$\alpha := \max_{\lambda \in [0,1]} \varphi(\lambda) := [f(\lambda y + (1 - \lambda)x) - (\lambda f(y) + (1 - \lambda) f(x))].$$

We need to show that $\alpha \leq 0$. Since $[0, 1]$ is compact and φ is continuous, there is a $\lambda \in [0, 1]$ such that $\varphi(\lambda) = \alpha$. If λ equals zero or one, we are done. Hence we may as well assume that $0 < \lambda < 1$ in which case

$$0 = \varphi'(\lambda) = \nabla f(x_\lambda)^T (y - x) + f(x) - f(y)$$

where $x_\lambda = x + \lambda(y - x)$, or equivalently

$$\lambda f(y) = \lambda f(x) - \nabla f(x_\lambda)^T (x - x_\lambda).$$

But then

$$\begin{aligned} \alpha &= f(x_\lambda) - (f(x) + \lambda(f(y) - f(x))) \\ &= f(x_\lambda) + \nabla f(x_\lambda)^T (x - x_\lambda) - f(x) \\ &\leq 0 \end{aligned}$$

by (b).

2) Suppose f is convex and let $x, d \in \mathbb{R}^n$, then by (b) of Part (1),

$$f(x + td) \geq f(x) + t \nabla f(x)^T d$$

for all $t \in \mathbb{R}$. Replacing the left hand side of this inequality with its second order Taylor expansion yields the inequality

$$f(x) + t \nabla f(x)^T d + \frac{t^2}{2} d^T \nabla^2 f(x) d + o(t^2) \geq f(x) + t \nabla f(x)^T d$$

or equivalently

$$\frac{1}{2} d^t \nabla^2 f(x) d + \frac{o(t^2)}{t^2} \geq 0.$$

Letting $t \to 0$ yields the inequality

$$d^T \nabla^2 f(x) d \geq 0.$$

Since d was arbitrary, $\nabla^2 f(x)$ is positive semi-definite.

Conversely, if $x, y \in \mathbb{R}^n$, then by the mean value theorem there is a $\lambda \in (0,1)$ such that

$$f(y) = f(x) + \nabla f(x)^T(y-x) + \frac{1}{2}(y-x)^T\nabla^2 f(x_\lambda)(y-x)$$

where $x_\lambda = \lambda y + (1-\lambda)_x$. Hence

$$f(y) \geq f(x) + \nabla f(x)^T(y-x)$$

since $\nabla^2 f(x_\lambda)$ is positive semi-definite. Therefore, f is convex by (b) of Part (1).

□

The final characterization of convexity in the above theorem is very useful when it applies. But it requires one to check when a matrix is positive definite. One approach to this is to compute the eigenvalues of the Hessian, but this can be a very time consuming procedure. For this reason, we give the following test for positive definiteness.

Theorem 1.12. *Let $H \in \mathbb{R}^{n\times n}$ be symmetric. We define the kth principal minor of H, denoted $\Delta_k(H)$, to be the determinant of the upper left-hand $k \times k$ submatrix of H. Then H is positive definite if and only if $\Delta_k(H) > 0$ for $k = 1, 2, \ldots, n$.*

Example
Consider the matrix

$$H = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 5 & 1 \\ -1 & 1 & 4 \end{bmatrix}.$$

We have

$$\Delta_1(H) = 1, \quad \Delta_2(H) = \begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix} = 4, \quad \Delta_3(H) = \det(H) = 8.$$

Therefore, H is positive definite.

1.4.1. *More on the Directional Derivative.* It is a surprising fact that convex function are directionally differentiable at every point of their domain in every direction. But this is just the beginning of the story. The directional derivative of a convex function possess several other important and surprising properties. We now develop a few of these.

Definition 1.4. *Let $h : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$. We say that h is positively homogeneous if*

$$h(\lambda x) = \lambda h(x) \quad \text{for all } x \in \mathbb{R} \text{ and } \lambda > 0.$$

We say that h is subadditive if

$$h(x+y) \leq h(x) + h(y) \quad \text{for all } x, y \in \mathbb{R}.$$

Finally, we say that h is sublinear if it is both subadditive and sublinear.

There are numerous important examples of sublinear functions (as we shall soon see), but perhaps the most familiar of these is the norm $\|x\|$. Positive homogeneity is obvious and subadditivity is simply the triangle inequality. In a certain sense the class of sublinear function is a generalization of norms. It is also important to note that sublinear functions are always convex functions. Indeed, given $x, y \in \mathrm{dom}\,(h)$ and $0 \leq \lambda \leq 1$,

$$\begin{aligned} h(\lambda x + (1-\lambda)y) &\leq h(\lambda x) + h(1-\lambda)y) \\ &= \lambda h(x) + (1-\lambda)h(y). \end{aligned}$$

Theorem 1.13. *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a convex function. Then at every point $x \in \mathrm{dom}\,(f)$ the directional derivative $f'(x;d)$ is a sublinear function of the d argument, that is, the function $f'(x;\cdot) : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is sublinear. Thus, in particular, the function $f'(x;\cdot)$ is a convex function.*

Proof. Let $x \in \mathrm{dom}\,(f)$, $d \in \mathbb{R}^n$, and $\lambda > 0$. Then

$$\begin{aligned} f'(x;\lambda d) &= \lim_{t \downarrow 0} \frac{f(x + t\lambda d) - f(x)}{t} \\ &= \lim_{t \downarrow 0} \lambda \frac{f(x + t\lambda d) - f(x)}{\lambda t} \\ &= \lambda \lim_{(\lambda t) \downarrow 0} \frac{f(x + (t\lambda) d) - f(x)}{(\lambda t)} \\ &= \lambda f'(x;d), \end{aligned}$$

showing that $f'(x;\cdot)$ is positively homogeneous.

Next let $d_1, d_2 \in \mathbb{R}^n$, Then

$$\begin{aligned} f'(x;d_1+d_2) &= \lim_{t \downarrow 0} \frac{f(x + t(d_1+d_2)) - f(x)}{t} \\ &= \lim_{t \downarrow 0} \frac{f(\frac{1}{2}(x + 2td_1) + \frac{1}{2}(x + 2td_2)) - f(x)}{t} \\ &\leq \lim_{t \downarrow 0} \frac{\frac{1}{2}f(x + 2td_1) + \frac{1}{2}f(x + 2td_2) - f(x)}{t} \\ &\leq \lim_{t \downarrow 0} \frac{\frac{1}{2}(f(x + 2td_1) - f(x)) + \frac{1}{2}(f(x + 2td_2) - f(x))}{t} \\ &= \lim_{t \downarrow 0} \frac{f(x + 2td_1) - f(x)}{2t} + \lim_{t \downarrow 0} \frac{f(x + 2td_2) - f(x)}{2t} \\ &= f'(x;d_1) + f'(x;d_2), \end{aligned}$$

showing that $f'(x;\cdot)$ is subadditive and completing the proof. ☐

Exercises

(1) Show that the functions

$$f(x_1, x_2) = x_1^2 + x_2^3, \quad \text{and} \quad g(x_1, x_2) = x_1^2 + x_2^4$$

both have a critical point at $(x_1, x_2) = (0, 0)$ with associated Hessian positive semi-definite. But $(0, 0)$ is only a local (global) minimizer for g and not for f.

(2) Find the local minimizers and maximizers for the following functions if they exist:
  (a) $f(x) = x^2 + \cos x$
  (b) $f(x_1, x_2) = x_1^2 - 4x_1 + 2x_2^2 + 7$
  (c) $f(x_1, x_2) = e^{-(x_1^2 + x_2^2)}$
  (d) $f(x_1, x_2, x_3) = (2x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - 1)^2$

(3) Which of the functions in problem 2 above are convex and why?

(4) If $f : \mathbb{R}^n \to \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is convex, show that the sets $\text{lev}_f(\alpha) = \{x : f(x) \le \alpha\}$ are convex sets for every $\alpha \in \mathbb{R}$. Let $h(x) = x^3$. Show that the sets $\text{lev}_h(\alpha)$ are convex for all α, but the function h is not itself a convex function.

(5) Show that each of the following functions is convex.
  (a) $f(x) = e^{-x}$
  (b) $f(x_1, x_2, \ldots, x_n) = e^{-(x_1 + x_2 + \cdots + x_n)}$
  (c) $f(x) = \|x\|$

(6) Consider the linear equation

$$Ax = b,$$

where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. When $n < m$ it is often the case that this equation is over-determined in the sense that no solution x exists. In such cases one often attempts to locate a 'best' solution in a least squares sense. That is one solves the *linear least squares problem*

$$(\text{lls}) : \text{ minimize } \frac{1}{2}\|Ax - b\|_2^2$$

for x. Define $f : \mathbb{R}^n \to \mathbb{R}$ by

$$f(x) := \frac{1}{2}\|Ax - b\|_2^2.$$

  (a) Show that f can be written as a quadratic function, i.e. a function of the form

$$f(x) := \frac{1}{2}x^T Q x - a^T x + \alpha\ .$$

  (b) What are $\nabla f(x)$ and $\nabla^2 f(x)$?
  (c) Show that $\nabla^2 f(x)$ is positive semi-definite.
  (d) * Show that a solution to (lls) must always exist.
  (e) * Provide a necessary and sufficient condition on the matrix A (**not on the matrix** $A^T A$) under which (lls) has a unique solution and then display this solution in terms of the data A and b.

(7) Consider the functions

$$f(x) = \frac{1}{2}x^T Qx - c^T x$$

and

$$f_t(x) = \frac{1}{2}x^T Qx - c^T x + t\phi(x),$$

where $t > 0$, $Q \in \mathbb{R}^{n\times n}$ is positive semi-definite, $c \in \mathbb{R}^n$, and $\phi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is given by

$$\phi(x) = \begin{cases} -\sum_{i=1}^n \ln x_i & , \text{ if } x_i > 0,\ i = 1, 2, \ldots, n, \\ +\infty & , \text{ otherwise.} \end{cases}$$

(a) Show that ϕ is a convex function.
(b) Show that both f and f_t are convex functions.
(c) Show that the solution to the problem $\min f_t(x)$ always exists and is unique.

(8) Classify each of the following functions as either coercive or non-coercive showing why you classification is correct.
(a) $f(x, y, z) = x^3 + y^3 + z^3 - xyz$
(b) $f(x, y, z) = x^4 + y^4 + z^2 - 3xy - z$
(c) $f(x, y, z) = x^4 + y^4 + z^2 - 7xyz^2$
(d) $f(x, y) = x^4 + y^4 - 2xy^2$
(e) $f(x, y, z) = \log(x^2y^2z^2) - x - y - z$
(f) $f(x, y, z) = x^2 + y^2 + z^2 - \sin(xyz)$

(9) Show that each of the following functions is convex or strictly convex.
(a) $f(x, y) = 5x^2 + 2xy + y^2 - x + 2y + 3$
(b) $f(x, y) = \begin{cases} (x + 2y + 1)^8 - \log((xy)^2), & \text{if } 0 < x,\ 0 < y, \\ +\infty, & \text{otherwise.} \end{cases}$
(c) $f(x, y) = 4e^{3x-y} + 5e^{x^2+y^2}$
(d) $f(x, y) = \begin{cases} x + \frac{2}{x} + 2y + \frac{4}{y}, & \text{if } 0 < x,\ 0 < y, \\ +\infty, & \text{otherwise.} \end{cases}$

(10) Compute the global minimizers of the functions given in the previous problem if they exist.