1.

$$\nabla^2 f(x_1, x_2) = \begin{pmatrix} 2 & 0 \\ 0 & 6x_2 \end{pmatrix}$$

is not positive semi-definite for $x_2 < 0$,so f is not convex thus no local (global) solution;
While

$$\nabla^2 g(x_1, x_2) = \begin{pmatrix} 2 & 0 \\ 0 & 12x_2^2 \end{pmatrix}$$

is positive semi-denifite for any $(x_1, x_2)^T$,hence function g is convex. It's critical point is local minimizer.

2. (a) $x = 0$ is local minimizer;
   (b) $(x_1, x_2)^T = (2, 0)^T$ is local minizer;
   (c) $(x_1, x_2) = (0, 0)$ is local maximizer;
   (d) $(x_1, x_2, x_3) = (\frac{1}{2}, 1, 1)$ is local minimizer.

3. (a),(b) and (d) are convex;(c) is concave, i.e. $-f(x_1, x_2)$ is convex.

4. For any $x_1, x_2 \in lev_f(\alpha)$ and $\lambda \in (0, 1)$,

$$\begin{aligned} f(\lambda x_1 + (1-\lambda)x_2) &\leq \lambda f(x_1) + (1-\lambda)f(x_2) \\ &\leq \lambda\alpha + (1-\lambda)\alpha \\ &= \alpha \end{aligned}$$

So $\lambda x_1 + (1-\lambda)x_2 \in lef_f(\alpha)$,hence $lev_f(\alpha)$ is a convex set.

$lev_f(\alpha) = \{x : x^3 \leq \alpha\} = (-\inf, \alpha^{\frac{1}{3}}]$ is an interval which is obviously a convex set.

But $h''(x) = 6x < 0$ for $x < 0$,so h is not convex function.

5. (a) $\nabla^2 f(x) = e^{-x} > 0$ for any x;
   (b)

$$\nabla^2 f(x_1, x_2, \cdots, x_n) = e^{-(x_1+x_2+\cdots+x_n)} \begin{pmatrix} 1 & \cdots & 1 \\ \cdots & & \cdots \\ 1 & \cdots & 1 \end{pmatrix}$$

   is positive semi-denifite for any $(x_1, x_2, \cdots, x_n)$.

   (c) for any x_1, x_2 and $\lambda \in (0, 1)$,by Triangular Inequality,

$$||\lambda x_1 + (1-\lambda)x_2)|| \leq ||\lambda x_1|| + ||(1-\lambda)x_2)|| = \lambda||x_1|| + (1-\lambda)||x_2||$$

   So by definition function $||x||$ is convex.

6. (a) $Q = A^T A$,$a^T = b^T A$,$\alpha = b^T b$;

(b) $\nabla f(x) = A^T Ax - A^T b$ and $\nabla^2 f(x) = A^T A$;

(c) For any $y \in R^n, y^T \nabla^2 f(x) y = (Ax)^T (Ax) = \|Ax\|_2^2 \geq 0$, so $\nabla^2 f(x)$ is positive semi-definite;

(d) There is at least one $\bar{x}$ satify $\nabla f(\bar{x}) = 0$,i.e,$A^T A\bar{x} = A^T b$ by linear algebra knowledge;

(e) A unique solution exits iff $Nul(A) = \{0\}$; under this condition,$A^T A$ is invertable and the unique solution is $\bar{x} = (A^T A)^{-1} A^T b$.