Solving LPs
The Simplex Algorithm of George Dantzig

We illustrate a general solution procedure, called the simplex algorithm, by implementing it on a very simple example. Consider the LP

$$
\begin{array}{rrcl}
(0.1) & \max 5x_1 + 4x_2 + 3x_3 & & \\
& \text{s.t. } 2x_1 + 3x_2 + x_3 & \leq & 5 \\
& 4x_1 + x_2 + 2x_3 & \leq & 11 \\
& 3x_1 + 4x_2 + 2x_3 & \leq & 8 \\
& 0 \leq x_1, x_2, x_3 & &
\end{array}
$$

The first step in our solution procedure is to rewrite the problem so that it looks more like the problem of solving a system of linear equations. We can then apply our knowledge of such systems to solving the LP (1.1). In order to do this we rewrite the linear inequality constraints as linear equations by introducing a new non-negative variable for each inequality:

$$
\begin{array}{rcll}
x_4 & = & 5 - [2x_1 + 3x_2 + x_3] & \geq 0, \\
x_5 & = & 11 - [4x_1 + x_2 + 2x_3 + x_5] & \geq 0, \\
x_6 & = & 8 - [3x_1 + 4x_2 + 2x_3 + x_6] & \geq 0.
\end{array}
$$

In addition, we introduce a new variable z for the objective value:

$$z = 5x_1 + 4x_2 + 3x_3.$$

Then all of the information associated with the LP (1.1) can be coded as follows:

$$
\begin{array}{rrcccccl}
 & 2x_1 + 3x_2 + x_3 & + & x_4 & & & = & 5 \\
 & 4x_1 + x_2 + 2x_3 & & + & x_5 & & = & 11 \\
(0.2) & 3x_2 + 4x_2 + 2x_3 & & & + & x_6 & = & 8 \\
 & -z + 5x_1 + 4x_2 + 3x_3 & & & & & = & 0 \\
 & & 0 \leq x_1, x_2, x_3, x_4, x_5, x_6. & & & & &
\end{array}
$$

The new variables x_4, x_5, and x_6 are called slack variables. This system can also be written in matrix notation as

$$
\begin{bmatrix} 0 & A & I \\ -1 & c^T & 0 \end{bmatrix} \begin{bmatrix} z \\ x \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix},
$$

where

$$
A = \begin{bmatrix} 2 & 3 & 1 \\ 4 & 1 & 2 \\ 3 & 4 & 2 \end{bmatrix}, \; I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; b = \begin{bmatrix} 5 \\ 11 \\ 8 \end{bmatrix}, \text{ and } c = \begin{bmatrix} 5 \\ 4 \\ 3 \end{bmatrix}.
$$

The associated augmented system is

$$\text{(0.3)} \qquad \begin{bmatrix} 0 & A & I & b \\ -1 & c & 0 & 0 \end{bmatrix}$$

and is referred to as a simplex tableau for the LP (1.1). Again consider the system

$$\text{(0.4)} \qquad \begin{aligned} x_4 &= 5 - 2x_1 - 3x_2 - x_3 \\ x_5 &= 11 - 4x_1 - x_2 - 2x_3 \\ x_6 &= 8 - 3x_1 - 4x_2 - 2x_3 \\ z &= 5x_1 + 4x_2 + 3x_3. \end{aligned}$$

In this system we are representing the variables x_4, x_5, x_6 and z as linear combinations of the variables x_1, x_2, and x_3. We call this system a dictionary for the LP (1.1) since it provides a definition for the objective value z and 3 of the variables (where 3 is the number of slack variables) in terms of the remaining variable. We call this the initial dictionary for (1.1). The variables that are "defined" in this way are called the basic variables, while the remaining variables are called nonbasic. The set of all basic variables is called the basis. A particular solution to this system is easily obtained by setting the nonbasic variables equal to zero. In this case, we get

$$\begin{aligned} x_4 &= 5 \\ x_5 &= 11 \\ x_6 &= 8 \\ z &= 0. \end{aligned}$$

Note that the solution

$$\text{(0.5)} \qquad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 5 \\ 4 \\ 8 \end{pmatrix}$$

is feasible for the extended system (1.2) since all components are non-negative. For this reason, we call the dictionary (1.4) a feasible dictionary for the LP (1.1).

The grand strategy of the simplex algorithm is to move from one feasible dictionary representation of the system (1.2) to another while simultaneously increasing the value of the objective variable z. In the current setting, beginning with the dictionary (1.4) how might one proceed?

First note that each feasible dictionary identifies one and only one feasible point obtained by setting all of the nonbasic variables equal to zero. This is how we obtain (1.5). To change the feasible point identified in this way, we need to increase the value of one of the non-basic variables. Note that we cannot decrease the value of a non-basic variable since we wish to remain feasible, that is, we wish to keep all variables non-negative. Also note that the coefficient of each of the non-basic variables in the representation of the objective value z in (1.4) is positive, hence when we increase the value of any one of these variables from zero, we automatically increase the value of the objective value z. Since the coefficient on x_1 in the representation of z is the greatest, we can increase z the fastest by increasing x_1.

By how much can we increase x_1 and still remain feasible? For example, if we increase x_1 to 3 then (1.4) says that $x_4 = -1$, $x_5 = -1$, $x_6 = -1$ which is not feasible. Note that x_4 remains non-negative as long as $x_1 \leq 5/2$, x_5 remaining non-negative if $x_1 \leq 11/4$, and x_6 remaining non-negative if $x_2 \leq 8/3$. Therefore, the most we can increase x_1 by and yet keep all variables non-negative is

$$\frac{5}{2} = \min\left\{\frac{5}{2}, \frac{11}{4}, \frac{8}{3}\right\}.$$

If we now increase x_2 to $\frac{5}{2}$, then the value of x_4 is driven to zero. Hence, we move x_1 into the basis and move x_4 out. We do this with the aid of the equation for x_4:

$$x_4 = 5 - 2x_1 - 3x_2 - x_3$$

converts to

$$2x_1 = 5 - x_4 - 3x_2 - x_3$$

or

$$x_1 = \frac{5}{2} - \frac{1}{2}x_4 - \frac{3}{2}x_2 - \frac{1}{2}x_3.$$

Substituting this expression for x_1 into the right hand side of (1.4) yields the new dictionary

$$\begin{aligned}
x_1 &= \frac{5}{2} - \frac{1}{2}x_4 - \frac{3}{2}x_2 - \frac{1}{2}x_3 \\
x_5 &= 11 - 4\left(\frac{5}{2} - \frac{1}{2}x_4 - \frac{3}{2}x_2 - \frac{1}{2}x_3\right) - x_2 - 2x_3 \\
&= 1 + 2x_4 + 5x_2 \\
x_6 &= 8 - 3\left(\frac{5}{2} - \frac{1}{2}x_4 - \frac{3}{2}x_2 - \frac{1}{2}x_3\right) - 4x_2 - 2x_3 \\
&= \frac{1}{2} + \frac{3}{2}x_4 + \frac{1}{2}x_2 - \frac{1}{2}x_3 \\
z &= 5\left(\frac{5}{2} - \frac{1}{2}x_4 - \frac{3}{2}x_2 - \frac{1}{2}x_3\right) + 4x_2 + 3x_3 \\
&= \frac{25}{2} - \frac{5}{2}x_4 - \frac{7}{2}x_2 + \frac{1}{2}x_3.
\end{aligned}$$

Recall that this process is simply Gaussian elimination and so can be performed in a matrix context on the simplex tableau (1.3). We return to this later in order to obtain a more efficient computational technique.

We now have a new dictionary

$$\begin{aligned} x_1 &= \frac{5}{2} - \frac{1}{2}x_4 - \frac{3}{2}x_2 - \frac{1}{2}x_3 \\ x_5 &= 1 + 2x_4 + 5x_2 \\ x_6 &= \frac{1}{2} + \frac{3}{2}x_4 + \frac{1}{2}x_2 - \frac{1}{2}x_3 \\ z &= \frac{25}{2} - \frac{5}{2}x_4 - \frac{7}{2}x_2 + \frac{1}{2}x_5 \end{aligned}$$

which identifies the feasible solution

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} \frac{5}{2} \\ 0 \\ 0 \\ 0 \\ 1 \\ \frac{1}{2} \end{pmatrix}$$

and the associated objective value $z = \frac{25}{2}$. Can we do better? Note that the coefficient of x_3 in the representation of z is positive, hence if we increase the value of x_3 from zero, we will increase the value of z. By how much can we increase the value of x_3 and yet keep all the remaining variables non-negative? As before, we can increase x_3 by at most

$$1 = \min\{{}^{(5/2)}/_{(1/2)}, {}^{(1/2)}/_{(1/2)}\}.$$

When we do this we drive x_6 to zero, so x_3 enters the basis and x_6 leaves:

$$\begin{aligned} \frac{1}{2}x_3 &= \frac{1}{2} + \frac{3}{2}x_4 + \frac{1}{2}x_2 - x_6 \\ x_3 &= 1 + 3x_4 + x_2 - 2x_6 \\ x_1 &= \frac{5}{2} - \frac{1}{2}x_4 - \frac{3}{2}x_2 - \frac{1}{2}[1 + 3x_4 + x_2 - 2x_6] \\ &= 2 - 2x_4 - 2x_2 + x_6 \\ x_5 &= 1 + 2x_4 + 5x_2 \\ z &= \frac{25}{2} - \frac{5}{2}x_4 - \frac{7}{2}x_2 + \frac{1}{2}[1 + 3x_4 + x_2 - 2x_6] \\ &= 13 - x_4 - 3x_2 - x_6 \end{aligned}$$

yielding the dictionary

$$\begin{aligned} x_3 &= 1 + 3x_4 + x_2 - 2x_6 \\ x_1 &= 2 - 2x_4 + 2x_2 + x_6 \\ x_5 &= 1 + 2x_4 + 2x_2 \\ z &= 13 - x_4 - 3x_2 - x - 6 \end{aligned}$$

which identifies the feasible solution

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

having adjective value $z = 13$.

Can we do better? NO! This solution is optimal! The coefficient on the variables in the cost row of the dictionary are all negative, so increasing their value will decrease the value of the objective.

Let us know review this process using matrix notation. we begin with the simplex tableau

	Pivot column ↓							r	
0	$\textcircled{2}$	3	1	1	0	0	5	$\textcircled{\frac{5}{2}}$	← Pivot row
0	4	1	2	0	1	0	11	$\frac{11}{4}$	
0	2	4	2	0	0	1	8	$\frac{8}{3}$	
-1	$\textcircled{5}$	4	3	0	0	0	0		

0	1	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{5}{2}$
0	0	-5	0	-2	1	0	1
0	0	$-\frac{1}{2}$	$\textcircled{\frac{1}{2}}$	$-\frac{3}{2}$	0	1	$\frac{1}{2}$
-1	0	$-\frac{7}{2}$	$\frac{1}{2}$	$-\frac{5}{2}$	0	0	$-\frac{25}{2}$

0	1	2	0	2	0	-1	2
0	0	-5	0	-2	1	0	1
0	0	-1	1	-3	0	2	1
-1	0	-3	0	-1	0	-1	-13

Notice that we have performed the exact same arithmetic operations but in the more efficient matrix format. Also observe that further streamlining is possible. Since we never really use the z column, ignore it in our computations. In the future, we will not write this column when we write the symplex tableaus. But it is useful to remember that it is really there and can always re-insert it whenever convenient.

We now give our formal definition for a dictionary associated with an LP in standard form:

$$\mathcal{P} \qquad \max_{Ax \le b, 0 \le x} c^T x \qquad , \text{where } A \in \mathbb{R}^{m \times m}$$

Let

$$(D_I) \qquad \begin{aligned} x_{n+i} &= b_i - \sum_{j=1}^{n} a_{ij} x_j \\ z &= \quad - \sum_{j=1}^{n} c_j x_j \end{aligned}$$

be the defining system for the slack variables x_{n+i}, $i = 1, \cdots, n$ and the objective variable z. A dictionary for $\mathcal{P}$ is any system of the form

$$(D_B) \qquad \begin{aligned} x_i &= \widehat{b}_i - \sum_{j \in N} \widehat{a}_{ij} x_j \qquad i \in B \\ z &= \widehat{z} - \sum_{j \in N} \widehat{c}_j x_j \end{aligned}$$

where B and N are index sets contained in the set of integers $\{1, \ldots, n+m\}$ satisfying

(1) B contains m elements,

(2) $B \cap N = \emptyset$

(3) $B \cup N = \{1, 2, \ldots, n+m\}$,

and such that the systems (D_I) and (D_B) have identical solution sets. The set $\{x_j : j \in B\}$ is said to be the basis associated with the dictionary (D_B) (sometimes we will refer to the index set B as the basis for the sake of simplicity), and the variables x_i, $i \in N$ are said to be the non-basic variables associated with this dictionary. the point identified by this dictionary is

$$\begin{aligned} x_i &= \widehat{b}_i \qquad & i \in B \\ x_j &= 0 \qquad & j \in N. \end{aligned}$$

The dictionary is said to be feasible if $0 \le \widehat{b}_i$ for $i \in N$, and it is said to be optimal if $\widehat{c}_j \le 0$ $j \in N$. If the dictionary is feasible, then the point

$$\begin{aligned} x_i &= \widehat{b}_i \qquad & i \in B \\ x_j &= 0 \qquad & j \in N, \end{aligned}$$

is said to be a basic feasible solution (BFS) for $\mathcal{P}$.

There are corresponding notions associated with the simplex tableau. Recall the initial augmented matrix associated with the system (D_I):

$$\begin{bmatrix} A & I & b \\ c^T & 0 & 0 \end{bmatrix}$$

(here we have dropped the z column). The matrix

$$\begin{bmatrix} \widehat{A} & R & \widehat{b} \\ \widehat{c}^T & -y^T & \widehat{z} \end{bmatrix}$$

is said to be a tableau for the linear program $\mathcal{P}$ if

(1) the matrix R (called the record matrix) is non-singular with

$$\begin{bmatrix} R & 0 \\ -y^T & 1 \end{bmatrix} \begin{bmatrix} A & I & b \\ c^T & 0 & 0 \end{bmatrix} = \begin{bmatrix} \widehat{A} & R & \widehat{b} \\ \widehat{c}^T & -y^T & \widehat{z} \end{bmatrix},$$

and

(2) the columns of the matrix $[\ \widehat{A} \quad R\ ]$ contain the m distinct columns of the $m \times m$ identity matrix.

The variables associated with the columns of the identity correspond to the basic variables. The tableau is said to be *primal feasible* if $\widehat{b} \geq 0$. It is said to be *dual feasible* if $\widehat{c} \leq 0$ and $0 \leq y$. It is said to be *optimal* if it is both primal and dual feasible.

Simplex Algorithm for Problems in Standard Form with Feasible Origin

Solve the following LPs using the simplex algorithm. All of the problems below are in standard form and have feasible origin.

1.
$$\begin{array}{lrcrcrcl} \text{maximize} & 4x & + & 3y & + & 2z & & \\ \\ \text{subject to} & x & & & + & z & \leq & 2 \\ & -x & - & y & + & z & \leq & 1 \\ & x & + & y & + & z & \leq & 3 \\ & 0 & \leq & x, & y, & z & & \end{array}$$

Solution: $(2, 1, 0)$, optimal value $= 11$.

2.
$$\begin{array}{lrcrcrcl} \text{maximize} & 4x & + & 2y & + & 2z & & \\ \\ \text{subject to} & x & + & 3y & - & 2z & \leq & 3 \\ & 4x & + & 2y & & & \leq & 4 \\ & x & + & y & + & z & \leq & 2 \\ & 0 & \leq & x, & y, & z & & \end{array}$$

Solution: $(1, 0, 1)$, optimal value $= 6$.

3.
$$\begin{array}{lrcrcrcl} \text{maximize} & -7x_1 & + & 9x_2 & + & 3x_3 & & \\ \\ \text{subject to} & 5x_1 & - & 4x_2 & - & x_3 & \leq & 10 \\ & x_1 & - & x_2 & & & \leq & 4 \\ & -3x_1 & + & 4x_2 & + & x_3 & \leq & 1 \\ & 0 & \leq & x_1, & x_2, & x_3. & & \end{array}$$

Solution: $(4, 0, 13)$, optimal value $= 11$.

4.

$$\begin{array}{lrcrcrcrcrcl}
\text{maximize} & 7x_1 & + & 6x_2 & + & 5x_3 & - & 2x_4 & + & 3x_5 & & \\
\\
\text{subject to} & x_1 & + & 3x_2 & + & 5x_3 & - & 2x_4 & + & 2x_5 & \leq & 4 \\
 & 4x_1 & + & 2x_2 & - & 2x_3 & + & x_4 & + & x_5 & \leq & 3 \\
 & 2x_1 & + & 4x_2 & + & 4x_3 & - & 2x_4 & + & 5x_5 & \leq & 5 \\
 & 3x_1 & + & x_2 & + & 2x_3 & - & x_4 & + & 2x_5 & \leq & 1 \\
 & & 0 & \leq & x_1, & x_2, & x_3, & x_4, & x_5. & & &
\end{array}$$

Solution: $(0, 4/3, 2/3, 5/3, 0)$, optimal value $= 8$.