Linear Programming Primer for Math 408

1 Introduction

1.1 What is Linear Programming?

A mathematical optimization problem is one in which some function is either maximized or minimized relative to a given set of alternatives. The function to be minimized or maximized is called the *objective function* and the set of alternatives is called the feasible region (or constraint region). In this course, the feasible region is always taken to be a subset of $\mathbb{R}^n$ (real n-dimensional space) and the objective function is a function from $\mathbb{R}^n$ to $\mathbb{R}$.

We further restrict the class of optimization problems that we consider to linear programming problems (or LPs). An LP is an optimization problem over $\mathbb{R}^n$ wherein the objective function is a linear function, that is, the objective has the form

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

for some $c_i \in \mathbb{R}$ $i = 1, \ldots, n$, and the feasible region is the set of solutions to a finite number of linear inequality and equality constraints, of the form

$$a_{i1}x_i + a_{i2}x_2 + \cdots + a_{in}x_n \leq b_i \qquad i = 1, \ldots, s$$

and

$$a_{i1}x_i + a_{i2}x_2 + \cdots + a_{in}x_n = b_i \qquad i = s+1, \ldots, m.$$

Linear programming is an extremely powerful tool for addressing a wide range of applied optimization problems. A short list of application areas is resource allocation, production scheduling, warehousing, layout, transportation scheduling, facility location, flight crew scheduling, parameter estimation, ...

1.2 An Example

To illustrate some of the basic features of LP, we begin with a simple two-dimensional example. In modeling this example, we will review the four basic steps in the development of an LP model:

1. Determine and label the *decision variables.*
2. Determine the objective and use the decision variables to write an expression for the *objective function.*
3. Determine the *explicit constraints* and write a functional expression for each of them.
4. Determine the *implicit constraints.*

PLASTIC CUP FACTORY

A local family-owned plastic cup manufacturer wants to optimize their production mix in order to maximize their profit. They produce personalized beer mugs and champaign glasses. The profit on a case of beer mugs is \$25 while the profit on a case of champaign glasses is \$20. The cups are manufactured with a machine called a plastic extruder which feeds on plastic resins. Each case of beer mugs requires 20 lbs. of plastic resins to produce while champaign glasses require 12 lbs. per case. The daily supply of plastic resins is limited to at most 1800 pounds. About 15 cases of either product can be produced per hour. At the moment the family wants to limit their work day to 8 hours.

We will model the problem of maximizing the profit for this company as an LP. The first step in our modeling process is to determine the *decision variables*. These are the variables that represent the quantifiable decisions that must be made in order to determine the daily production schedule. That is, we need to specify those quantities whose values completely determine a production schedule and its associated profit. In order to determine these quantities, one can ask the question "If I were the plant manager for this factory, what must I know in order to implement a production schedule?" The best way to determine the decision variables is to put oneself in the shoes of the decision maker and then ask the question "What do I need to know in order to make this thing work?" In the case of the plastic cup factory, everything is determined once it is known how many cases of beer mugs and champaign glasses are to be produced each day.

Decision Variables:

$B = \#$ of cases of beer mugs to be produced daily.

$C = \#$ of cases of champaign glasses to be produced daily.

You will soon discover that the most difficult part of any modeling problem is the determination of decision variables. Once these variables are correctly determined then the remainder of the modeling process usually goes smoothly.

After specifying the decision variables, one can now specify the problem objective. That is, one can write an expression for the objective function.

Objective Function:

Maximize profit where profit $= 25B + 20C$

The next step in the modeling process is to express the feasible region as the solution set of a finite collection of linear inequality and equality constraints. We separate this process into two steps:

1. determine the explicit constraints, and
2. determine the implicit constraints.

The explicit constraints are those that are explicitly given in the problem statement. In the problem under consideration, there are explicit constraints on the amount of resin and the number of work hours that are available on a daily basis.

Explicit Constraints:

resin constraint: $20B + 12C \leq 1800$

work hours constraint: $\frac{1}{15}B + \frac{1}{15}C \leq 8$.

This problem also has other constraints called implicit constraints. These are constraints that are not explicitly given in the problem statement but are present nonetheless. Typically these constraints are associated with "natural" or "common sense" restrictions on the decision variable. In the cup factory problem it is clear that one cannot have negative cases of beer mugs and champaign glasses. That is, both B and C must be non-negative quantities.

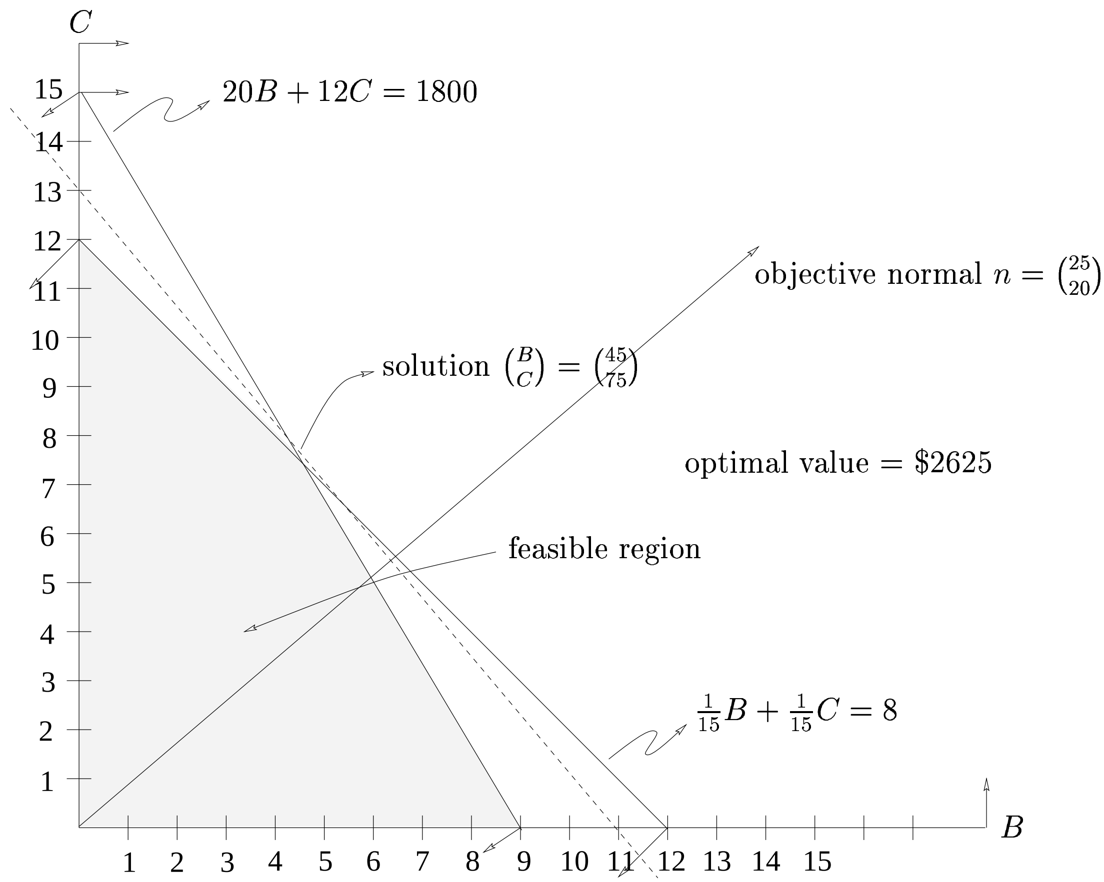
Implicit Constraints:

$$0 \leq B, \qquad 0 \leq C.$$

The entire model for the cup factory problem can now be succinctly stated as

$$\begin{array}{rl} \mathcal{P} : & \max 25B + 20C \\ & \text{subject to } \; 20B + 12C \;\leq\; 1800 \\ & \qquad \frac{1}{15}B + \frac{1}{15}C \;\leq\; 8 \\ & \qquad\qquad\qquad 0 \;\leq\; B, C \end{array}$$

Since this problem is two dimensional it is possible to provide a graphical solution. The first step toward a graphical solution is to graph the feasible region. To do this, first graph



the line associated with each of the linear inequality constraints. Then determine on which side of each of these lines the feasible region must lie (don't forget the implicit constraints!). Once the correct side is determined it is helpful to put little arrows on the line to remind yourself of the correct side. Then shade in the resulting feasible region.

The next step is to draw in the vector representing the gradient of the objective function at the origin. Since the objective function has the form

$$f(x_1, x_2) = c_1 x_1 + c_2 x_2,$$

the gradient of f is the same at every point in $\mathbb{R}^2$;

$$\nabla f(x_1, x_2) = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

Recall from calculus that the gradient always points in the direction of increasing function values. Moreover, since the gradient is constant on the whole space, the level sets of f associated with different function values are given by the lines perpendicular to the gradient. Consequently, to obtain the location of the point at which the objective is maximized we simply set a ruler perpendicular to the gradient and then move the ruler in the direction of the gradient until we reach the last point (or points) at which the line determined by the ruler intersects the feasible region. In the case of the cup factory problem this gives the solution to the LP as $\binom{B}{C} = \binom{45}{75}$

We now recap the steps followed in the solution procedure given above:

Step 1: Graph each of the linear constraints indication on which side of the constraint the feasible region must lie. Don't forget the implicit constraints!

Step 2: Shade in the feasible region.

Step 3: Draw the gradient vector of the objective function.

Step 4: Place a straightedge perpendicular to the gradient vector and move the straightedge either in the direction of the gradient vector for maximization, or in the opposite direction of the gradient vector for minimization to the last point for which the straightedge intersects the feasible region. The set of points of intersection between the straightedge and the feasible region is the set of solutions to the LP.

The solution procedure described above for two dimensional problems reveals a great deal about the geometric structure of LPs that remains true in n dimensions. We will explore this geometric structure more fully as the course evolves. But for the moment, we continue to study this 2 dimensional LP to see what else can be revealed about the structure of this problem.

Before leaving this section, we make a final comment on the modeling process described above. We emphasize that there is not one and only one way to model the Cup Factory problem, or any problem for that matter. In particular, there are many ways to choose the decision variables for this problem. Clearly, it is sufficient for the shop manager to know how many hours each days should be devoted to the manufacture of beer mugs and how many hours to champaign glasses. From this information everything else can be determined. For example, the number of cases of beer mugs that get produced is 15 times the number of hours devoted to the production of beer mugs. Therefore, as you can see there are many ways to model a given problem. But in the end, they should all point to the same optimal process.

1.3 Duality Theory

We now briefly discuss how the "hidden hand of the market place" gives rise to a theory of dual linear programs. Think of the cup factory production process as a black box through which the resources flow. Raw resources go in one end and exit the other. When they come out the resources have a different form, but whatever comes out is still comprised of the entering resources. However, something has happened to the value of the resources by passing through the black box. The resources have been purchased for one price as they enter the box and are sold in their new form as they leave. The difference between the entering and exiting prices is called the profit. Assuming that there is a positive profit the resources have increased in value as they pass through the production process. The marginal value of a resource is precisely the increase in the per unit value of the resource due to the production process.

Let us now consider how the market introduces pressures on the profitability and the value of the resources available to the market place. We take the perspective of the cup factory *vs* the market place. The market place does not want the cup factory to go out of business. On the other hand, it does not want the cup factory to see a profit. It wants to keep all the profit for itself and only let the cup factory just break even. It does this by setting the price of the resources available in the market place. That is, the market sets the price for plastic resin and labor and it tries to do so in such a way that the cup factory sees no profit and just breaks even. Since the cup factory is now seeing a profit, the market must figure out by how much the sale price of resin and labor must be raised to reduce this profit to zero. This is done by minimizing the value of the available resources over all price increments that guarantee that the cup factory either loses money or sees no profit from both of its products. If we denote the per unit price increment for resin by R and that for labor by L, then the profit for beer mugs is eliminated as long as

$$20R + \frac{1}{15}L \geq 25$$

since the left hand side represents the increased value of the resources consumed in the production of one case of beer mugs and the right hand side is the current profit on a case of beer mugs. Similarly, for champaign glasses, the market wants to choose R and L so that

$$12R + \frac{1}{15}L \geq 20.$$

Now in order to maintain equilibrium in the market place, that is, not drive the cup factory out of business (since then the market realizes no profit at all), the market chooses R and L so as to minimize the increased value of the available resources. That is, the market chooses R and L to solve the problem

$$\begin{array}{rlrcl} \mathcal{D} : & \text{minimize}\, 1800R + 8L & & & \\ & \text{subject to} & 20R + \frac{1}{15}L & \geq & 25 \\ & & 12R + \frac{1}{15}L & \geq & 20 \\ & & 0 & \leq & R, L \end{array}$$

This is just another LP. It is called the "dual" to the LP $\mathcal{P}$ in which the cup factory tries to maximize profit. Observe that if $\binom{B}{C}$ is feasible for $\mathcal{P}$ and $\binom{R}{L}$ is feasible for $\mathcal{D}$, then

$$\begin{array}{rcl} 25B + 20C & \leq & [20R + \frac{1}{15}L]B + [12R + \frac{1}{15}L]C \\ & = & R[20B + 12C] + L[\frac{1}{15}B + \frac{1}{15}C] \\ & \leq & 1800R + 8L. \end{array}$$

Thus, the value of the objective in $\mathcal{P}$ at a feasible point in $\mathcal{P}$ is bounded above by the objective in $\mathcal{D}$ at any feasible point for $\mathcal{D}$. In particular, the optimal value in $\mathcal{P}$ is bounded above by the optimal value in $\mathcal{D}$. The "strong duality theorem" states that if either of these problems has a finite optimal value, then so does the other and these values coincide. In addition, we claim that the solution to $\mathcal{D}$ is given by the marginal values for $\mathcal{P}$. That is, $\binom{R}{L} = \begin{bmatrix} 5/8 \\ 375/2 \end{bmatrix}$ is the optimal solution for $\mathcal{D}$. In order to show this we need only show that $\binom{R}{L} = \begin{bmatrix} 5/8 \\ 375/2 \end{bmatrix}$ is feasible for $\mathcal{D}$ and that the value of the objective in $\mathcal{D}$ at $\binom{R}{L} = \begin{bmatrix} 5/8 \\ 375/2 \end{bmatrix}$ coincides with the value of the objective in $\mathcal{P}$ at $\binom{B}{C} = \binom{45}{75}$. First we check feasibility:

$$0 \le \frac{5}{8}, \qquad 0 \le \frac{375}{2}$$

$$20 \cdot \frac{5}{8} + \frac{1}{15} \cdot \frac{375}{2} \ge 25$$

$$12 \cdot \frac{5}{8} + \frac{1}{15} \cdot \frac{375}{2} \ge 20.$$

Next we check optimality

$$25 \cdot 45 + 20 \cdot 75 = 2625 = 1800 \cdot \frac{5}{8} + 8 \cdot \frac{375}{2}.$$

1.4 LPs in Standard Form and Their Duals

Recall that a linear program is a problem of maximization or minimization of a linear function subject to a finite number of linear inequality and equality constraints. This general definition leads to an enormous variety of possible formulations. In this section we propose one fixed formulation for the purposes of developing an algorithmic solution procedure. We then show that every LP can be recast in this form. We say that an LP is in *standard form* if it has the form

$$\begin{array}{lll} \mathcal{P}: & \text{maximize} & c_1x_1 + c_2x_2 + \cdots + c_nx_n \\ & \text{subject to} & a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \;\le\; b_i \text{ for } i = 1, 2, \ldots, m \\ & & 0 \le x_j \text{ for } j = 1, 2, \ldots, n\ . \end{array}$$

Using matrix notation, we can rewrite this LP as

$$\begin{array}{lll} \mathcal{P}: & \text{maximize} & c^T x \\ & \text{subject to} & Ax \le b \\ & & 0 \le x\ , \end{array}$$

where the inequalities $Ax \le b$ and $0 \le x$ are to be interpreted componentwise.

Following the results of the previous section on LP duality, we claim that the dual LP to $\mathcal{P}$ is the LP

$$\begin{array}{lll} \mathcal{D}: & \text{minimize} & b_1y_1 + b_2y_2 + \cdots + b_ny_m \\ & \text{subject to} & a_{1j}y_1 + a_{2j}y_2 + \cdots + a_{mj}y_m \;\ge\; c_j \text{ for } j = 1, 2, \ldots, n \\ & & 0 \le y_i \text{ for } i = 1, 2, \ldots, m\ . \end{array}$$

Again, the statement of this $\mathcal{D}$ can be simplified by the use of matrix notation to give the problem

$$\begin{array}{lll} \mathcal{D}: & \text{minimize} & b^T y \\ & \text{subject to} & A^T y \geq c \\ & & 0 \leq y \ . \end{array}$$

Just as for the cup factory problem, the LPs $\mathcal{P}$ and $\mathcal{D}$ are related via the *Weak Duality Theorem.*

THEOREM: [WEAK DUALITY] *If $x \in \mathbb{R}^n$ is feasible for $\mathcal{P}$ and $y \in \mathbb{R}^m$ is feasible for $\mathcal{D}$, then*

$$c^T x \leq y^T A x \leq b^T y.$$

Thus, if $\mathcal{P}$ is unbounded, then $\mathcal{D}$ is infeasible, and if $\mathcal{D}$ is unbounded, then $\mathcal{P}$ is infeasible.

PROOF: Let $x \in \mathbb{R}^n$ be feasible for $\mathcal{P}$ and $y \in \mathbb{R}^m$ be feasible for $\mathcal{D}$. Then

$$\begin{array}{rcll} c^T x & = & \sum_{j=1}^{n} c_j x_j & \\ & \leq & \sum_{j=1}^{n} (\sum_{i=1}^{m} a_{ij} y_i) x_j & [\text{since } x_j \geq 0 \text{ and } \sum_{i=1}^{m} a_{ij} y_i \geq c_j] \\ & = & y^T A x & \\ & = & \sum_{i=1}^{m} (\sum_{j=1}^{n} a_{ij} x_j) y_i & \\ & \leq & \sum_{i=1}^{m} b_i y_i & [\text{since } y_i \geq 0 \text{ and } \sum_{j=1}^{n} a_{ij} x_j \leq b_i] \\ & = & b^T y & \end{array}$$

■

We caution that the infeasibility of either $\mathcal{P}$ or $\mathcal{D}$ does not imply the unboundedness of the other. Indeed, it is possible for both $\mathcal{P}$ and $\mathcal{D}$ to be infeasible as is illustrated by the following example.

EXAMPLE:

$$\begin{array}{lrcrcr} \text{maximize} & 2x_1 & - & x_2 & & \\ & x_1 & - & x_2 & \leq & 1 \\ & -x_1 & + & x_2 & \leq & -2 \\ & 0 & \leq & x_1, & x_2 & \end{array}$$

The Weak Duality Theorem yields the following elementary corollary.

COROLLARY 1.1 *Let $\bar{x}$ be feasible for $\mathcal{P}$ and $\bar{y}$ feasible for $\mathcal{D}$ if $c^T \bar{x} = b^T \bar{y}$, then $\bar{x}$ solves $\mathcal{P}$ and $\bar{y}$ solves $\mathcal{D}$.*

PROOF: Let x be any other vector feasible for $\mathcal{P}$. Then, by the WDT,

$$c^T x \leq b^T \bar{y} = c^T \bar{x}.$$

Therefore,

$$\begin{array}{lll} \text{maximize} & c^T x & \leq \ c^T \bar{x} \\ \text{subject to} & Ax \leq b, \ 0 \leq x & \end{array}$$

But $A\bar{x} \leq b$, $0 \leq \bar{x}$, so $\bar{x}$ solves $\mathcal{P}$. Similarly, if y is any other vector feasible for $\mathcal{D}$, then

$$b^T \bar{y} = c^T \bar{x} \leq b^T y.$$

Therefore

$$\begin{array}{lll} b^T \bar{y} \ \leq & \text{minimize} & b^T y \\ & \text{subject to} & A^T y \geq c, \ 0 \leq y, \end{array}$$

so that $\bar{y}$ solves $\mathcal{D}$. ■

THEOREM 1.1 (THE STRONG DUALITY THEOREM) *If either $\mathcal{P}$ or $\mathcal{D}$ has a finite optimal value, then so does the other and these optimal values coincide, and, in addition, optimal solutions to both $\mathcal{P}$ and $\mathcal{D}$ exist.*

Observe that this result states that the finiteness of the optimal value implies the existence of a solution. This is not always the case for nonlinear optimization problems. Indeed, consider the problem

$$\min_{x \in \mathbb{R}} e^x.$$

This problem has a finite optimal value, namely zero; however, this value is not attained by any point $x \in \mathbb{R}$. That is, it has a finite optimal value, but a solution does not exist. The existence of solutions when the optimal value is finite is one of the many special properties of linear programs.

1.4.1 Transformation to Standard Form

Every LP can be transformed to an LP in standard form. This process usually requires a transformation of variables and occasionally the addition of new variables. In this section we provide a step-by-step procedure for transforming any LP to one in standard form.

minimization $\rightarrow$ maximization

To transform a minimization problem to a maximization problem just multiply the objective function by -1.

linear inequalities

If an LP has an equality constraint of the form

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \geq b_i.$$

This inequality can be transformed to one in standard form by multiplying the inequality through by -1 to get

$$-a_{i1}x_1 - a_{i2}x_2 - \cdots - a_{in}x_n \leq -b_i.$$

linear equation

The linear equation

$$a_{i1}x_i + \cdots + a_{in}x_n = b_i$$

can be written as two linear inequalities

$$a_{i1}x_1 + \cdots + a_{in}x_n \leq b_i$$

and

$$a_{i1}x_1 + \cdots + a_{in}x_n \geq b_i.$$

The second of these inequalities can be transformed to standard form by multiplying through by -1.

variables with lower bounds

If a variable x_i has lower bound l_i which is not zero ($l_i \leq x_i$), one obtains a non-negative variable w_i with the substitution

$$x_i = w_i + l_i.$$

In this case, the bound $l_i \leq x_i$ is equivalent to the bound $0 \leq w_i$.

variables with upper bounds

If a variable x_i has an upper bound u_i ($x_i \leq u_i$) one obtains a non-negative variable w_i with the substitution

$$x_i = u_i - w_i.$$

In this case, the bound $x_i \leq u_i$ is equivalent to the bound $0 \leq w_i$.

variables with interval bounds

An interval bound of the form $l_i \leq x_i \leq u_i$ can be transformed into one non-negativity constraint and one linear inequality constraint in standard form by making the substitution

$$x_i = w_i + l_i.$$

In this case, the bounds $l_i \leq x_i \leq u_i$ are equivalent to the constraints

$$0 \leq w_i \qquad \text{and} \qquad w_i \leq u_i - l_i.$$

free variables

Sometimes a variable is given without any bounds. Such variables are called free variables. To obtain standard form every free variable must be replaced by the difference of two non-negative variables. That is, if x_i is free, then we get

$$x_i = u_i - v_i$$

with $0 \leq u_i$ and $0 \leq v_i$.

To illustrate the ideas given above, we put the following LP into standard form.

$$\begin{array}{lrcrcrcrcr} \text{minimize} & 3x_1 & - & x_2 & & & & & & \\ \text{subject to} & -x_1 & + & 6x_2 & - & x_3 & + & x_4 & \geq & -3 \\ & & & 7x_2 & & & + & x_4 & = & 5 \\ & & & & & x_3 & + & x_4 & \leq & 2 \end{array}$$

$$-1 \leq x_2, x_3 \leq 5, -2 \leq x_4 \leq 2.$$

First, we rewrite the objective as

$$\text{maximize } -3x_1 + x_2.$$

Next we replace the first inequality constraint by the constraint

$$x_1 - 6x_2 + x_3 - x_4 \leq 3.$$

The equality constraint is replaced by the two inequality constraints

$$\begin{array}{rcrcr} 7x_2 & + & x_4 & \leq & 5 \\ -7x_2 & - & x_4 & \leq & -5. \end{array}$$

Observe that the variable x_1 is free, so we replace it by

$$x_1 = y_1 - y_2 \text{ with } 0 \leq y_1, 0 \leq y_2.$$

The variable x_2 has a non-zero lower bound so we replace it by

$$x_2 = y_3 - 1 \text{ with } 0 \leq y_3.$$

The variable x_3 is bounded above, so we replace it by

$$x_3 = 5 - y_4 \text{ with } 0 \leq y_4.$$

The variable x_4 is bounded below and above so we replace it by

$$x_4 = y_5 - 2 \text{ with } 0 \leq y_5 \text{ and } y_5 \leq 4.$$

Putting it all together we get the following LP in standard form:

$$\begin{array}{lrcrcrcrcrcr}
\text{maximize} & -3y_1 & + & 3y_2 & + & y_3 & & & & & & \\
\text{subject to} & y_1 & - & y_2 & - & 6y_3 & - & y_4 & - & y_5 & \leq & -10 \\
 & & & & & 7y_3 & & & + & y_5 & \leq & 14 \\
 & & & & - & 7y_3 & & & - & y_5 & \leq & -14 \\
 & & & & & & - & y_4 & + & y_5 & \leq & -1 \\
 & & & & & & & & & y_5 & \leq & 4
\end{array}$$

$$0 \leq y_1, y_2, y_3, y_4, y_5.$$

Exercises

Graphical solutions of two dimensional LPs

1. Sketch the graph of the constraint region for the following LP's. Then solve them by sketching the optimal level set of the objective function.

$$\begin{array}{lrcrcr} \text{maximize} & 2x & + & 3y & & \\ \text{subject to} & -3x & + & y & \leq & 2 \\ & 4x & + & 2y & \leq & 44 \\ & 4x & - & y & \leq & 20 \\ & -x & + & 2y & \leq & 14 \\ & 0 & \leq & x, & y & \end{array}$$

$$\begin{array}{lrcrcr} \text{minimize} & x & + & y & & \\ \text{subject to} & -x & + & y & \leq & 3 \\ & 2x & + & y & \leq & 18 \\ & & & y & \geq & 6 \\ & 0 & \leq & x, & y & \end{array}$$

2. Graph the following function of α by graphically solving the necessary LPs:

$$\begin{array}{llrcrcr} v(\alpha) := & \text{maximize} & x_1 & + & \alpha x_2 & & \\ & \text{subject to} & x_1 & - & x_2 & \leq & 4 \\ & & x_1 & + & x_2 & \leq & 6 \\ & & -4x_1 & + & x_2 & \leq & -8 \\ & & 0 \leq & x_1, & x_2 & . & \end{array}$$

Exerices

Transformation of LPs to Standard Form

Transform the following LPs to LPs in standard form.

1.

$$\begin{array}{lrcrcrcr} \text{minimize} & x_1 & - & 12x_2 & - & 2x_3 & & \\ \\ \text{subject to} & 5x_1 & - & x_2 & - & 2x_3 & = & 10 \\ & 2x_1 & + & x_2 & - & 20x_3 & \geq & -30 \\ & x_2 \leq & 0 & , & 1 \leq & x_3 & \leq 4 & \end{array}$$

2.

$$\begin{array}{lrcrcrcr} \text{maximize} & 3x & - & 12y & + & 4z & & \\ \\ \text{subject to} & 5x & & & - & 10z & = & 10 \\ & 2x & - & y & - & 17z & \geq & -10 \\ & x & + & y & + & z & \leq & 10 \\ & y & \leq & 0, & 1 & \leq & z & \end{array}$$

3.

$$\begin{array}{lrcrcrcr} \text{minimize} & 4x_1 & - & 2x_2 & + & x_3 & & \\ & -x_1 & + & 3x_2 & - & x_3 & \geq & -1 \\ & & & 5x_2 & + & 3x_3 & = & 5 \\ & x_1 & + & x_2 & + & x_3 & \leq & 1 \\ & -1 & \leq & x_2, & -2 & \leq & x_3 & \leq 2 \end{array}$$