Computing the Dual of a General LP

Consider the general LP

$$\begin{array}{lll} \mathcal{P}_0 & \text{maximize} & c^T u + p^T v \\ & \text{subject to} & Au + Bv \leq r \\ & & Eu + Fv = h \\ & & 0 \leq u \ , \end{array}$$

where

$$A \in I\!R^{m\times n}, \quad B \in I\!R^{m\times t}, \quad E \in I\!R^{s\times n}, \quad \text{and} \quad F \in I\!R^{s\times t},$$

with

$$c \in I\!R^{n}, \quad p \in I\!R^{t}, \quad r \in I\!R^{m}, \quad \text{and} \quad h \in I\!R^{s}.$$

We wish to compute the dual to this LP. There are 4 general rules for computing the dual. They are as follows:

1. Linear inequality constraints such as
$$Au + Bv \leq r,$$
where $r \in I\!R^m$, give rise to a non-negative dual variable $y \in I\!R^m_+$.
2. Linear equality constraints such as
$$Eu + Fv = h,$$
where $h \in I\!R^s$, give rise to an unconstrained, or *free*, dual variable $w \in I\!R^s$.
3. A non-negatively constrained primal variable $u \in I\!R^n_+$ gives rise to linear inequality constraints on the dual variables using the columns of the primal constraint matrices associated with the primal variable u. In the case of $\mathcal{P}$, this gives the dual inequality constraint
$$A^T y + E^T w \geq c.$$
4. A *free*, or unconstrained, primal variable $v \in I\!R^t$ gives rise to linear equality constraints on the dual variables using the columns of the primal constraint matrices associated with the primal variable v. In the case of $\mathcal{P}$, this gives the dual equality constraint
$$B^T y + F^T w = p.$$

Therefore, the dual to $\mathcal{P}$ is

$$\begin{array}{lll} \mathcal{D}_0 & \text{minimize} & r^T y + h^T w \\ & \text{subject to} & A^T y + E^T w \geq c \\ & & B^T y + F^T w = p \\ & & 0 \leq y \ . \end{array}$$

As an application of this process we compute the dual to the following LP:

$$\begin{array}{lll} \mathcal{P}_1 & \text{maximize} & 3x_1 + 2x_2 + 5x_3 \\ & \text{subject to} & 5x_1 + 3x_2 + x_3 = -8 \\ & & 4x_1 + 2x_2 + 8x_3 \leq 23 \\ & & 6x_1 + 7x_2 + 3x_3 \geq 1 \\ & & x_1 \leq 4,\ 0 \leq x_3 . \end{array}$$

By rule 2 above, we get a free dual variable y_1 associate with the primal constraint

$$5x_1 + 3x_2 + x_3 = -8.$$

By rule 1 above, we get three non-negative dual variables y_2, y_3, and y_4 associated with the primal constraints

$$4x_1 + 2x_2 + 8x_3 \leq 23 ,$$

$$-6x_1 - 7x_2 - 3x_3 \leq -1 ,$$

and

$$x_1 \leq 4 ,$$

respectively. By rule 4, we get the dual linear equality constraints

$$\begin{array}{rcl} 5y_1 + 4y_2 - 6y_3 + y_4 & = & 3 \\ 3y_1 + 2y_2 - 7y_3 & = & 2 . \end{array}$$

By rule 3, we get the dual linear inequality constraint

$$y_1 + 8y_2 - 3y_3 \geq 5 .$$

Putting all of this together yields the dual problem

$$\begin{array}{lll} \mathcal{D}_1 & \text{minimize} & -8y_1 + 23y_2 - y_3 + 4y_4 \\ & & 5y_1 + 4y_2 - 6y_3 + y_4 = 3 \\ & & 3y_1 + 2y_2 - 7y_3 = 2 \\ & & y_1 + 8y_2 - 3y_3 \geq 5 \\ & & 0 \leq y_2,\ y_3,\ y_4 . \end{array}$$

Another example, on a more abstract level, is to compute the dual of the LP

$$\begin{array}{lll} \mathcal{P}_2 & \text{minimize} & c^T x \\ & \text{subject to} & Ax \leq b \text{ and } Ex = r . \end{array}$$

Since all of the primal variables are free, the dual has only equality linear constraints. by rule 4. The linear inequality constraint $Ax \leq b$ gives rise to a non-negative dual variable y, while the linear equality constraint $Ex = r$ gives rise to a free dual variable w. Hence the dual is

$$\begin{array}{lll} \mathcal{D}_2 & \text{maximize} & b^T y + r^T w \\ & \text{subject to} & A^T y + E^T w = -c \text{ and } 0 \leq y . \end{array}$$

(Where did the $-c$ come from?)

Exercises

Computing Dual LPs without Conversion to Standard Form

1. Compute the dual LP to each of the following LPs without first converting to standard form.

   (a)

$$\begin{array}{ll} \text{maximize} & 2x_1 - 3x_2 + 10x_3 \\ \text{subject to} & x_1 + x_2 - x_3 = 12 \\ & x_1 - x_2 + x_3 \leq 8 \\ & 0 \leq x_2 \leq 10 \end{array}$$

   (b)

$$\begin{array}{llllll} \text{maximize} & & 42x_2 & -30x_3 & & \\ \text{subject to} & x_1 & -x_2 & +x_3 & -x_4 & = 0 \\ & x_1 & & +x_3 & -x_4 & \leq 5 \\ & & 5x_2 & +x_3 & -5x_4 & = -1 \\ & 0 & \leq x_1, & & 0 \leq & x_3 \leq 20 \end{array}$$

2. Consider the mini–max problem

$$\min_{x \in I\!R^n} \max_{i=1,2,\ldots,m} \{a_i^T x - b_i\}$$

   where $a_i \in I\!R^n$ and $b_i \in I\!R$ for $i = 1, 2, \ldots, m$.

   (a) Show that this mini-max problem is in some sense *equivalent* to the LP

(1)
$$\begin{array}{ll} \text{maximize} & -x_0 \\ \text{subject to} & Ax - b \leq x_0 e, \end{array}$$

   where $A = (a_{ij})_{m \times n}$, $b = [b_1, b_2, \ldots, b_m]^T$, and $e \in I\!R^m$ is the vector of all ones.

   (b) Show that the dual of the LP (1) is

$$\begin{array}{ll} \text{minimize} & b^T y \\ \text{subject to} & A^T y = 0, \; e^T y = 1, \\ & 0 \leq y \end{array}$$

3. Consider the system of linear inequalities and equations

(2)
$$Ax \leq b, \qquad Bx = d,$$

   where $A \in I\!R^{m \times n}$, $B \in I\!R^{s \times t}$, $d \in I\!R^s$, and $b \in I\!R^n$. We are interested in studying the consistency of this system, that is, we are interested in determining conditions under which the solution set $S = \{x : Ax \leq b,\ Bx = d\}$ is non–empty. For this purpose, we make use of the following linear program:

$$\begin{array}{lrl} \mathcal{P}: \text{ maximize} & -e^T z & \\ & Ax - z & \leq b \\ & Bx & = d \\ & 0 & \leq z \end{array}$$

where $e \in I\!R^m$ is the vector of all ones ($e = (1, 1, 1, \ldots, 1)^T$).

(a) Show that the system (2) is consistent (i.e. $S \neq \emptyset$) if and only if the optimal value in $\mathcal{P}$ is zero.

(b) Show that the dual to the LP $\mathcal{P}$ is the LP

$$\begin{array}{ll} \mathcal{D}: \text{ minimize} & b^T u + d^T v \\ & A^T u + B^T v = 0 \\ & 0 \leq u \leq e. \end{array}$$

(c) Show that the system $Ax \leq b,\ Bx = d$ is inconsistent (i.e. $S = \emptyset$) if and only if there are vectors $u \in I\!R^m$ and $v \in I\!R^s$ such that $0 \leq u,\ A^T u + B^T v = 0$, and $b^T u + d^T v < 0$.