MATH 408 SAMPLE FINAL EXAM SOLUTIONS

1. *Fixed Income Securities: Bonds*

Fifteen years ago today Mary purchased a 10% bond with 30 years to maturity and a face value of \$10,000. Assume the yield at the time of purchase was 10% and the yield today is 3%. Also assume that the coupon payments occur only once a year.

I expect you to compute a numerical value for each question. Since calculators are not allowed, you may use the following helpful numbers:

$$\left(\frac{1}{1.1}\right)^{30} = 0.057, \quad \left(\frac{1}{1.1}\right)^{15} = 0.24, \quad \left(\frac{1}{1.03}\right)^{30} = 0.41, \quad \left(\frac{1}{1.03}\right)^{15} = 0.64$$

(a)(20 points) At what price did Mary purchase the bond 15 years ago?

SOLUTION: The bond was at par when purchased, so the sale price was \$10,000.

(b)(40 points) At what price can she sell the bond today?

SOLUTION: The bond price formula states that the price today is

$$\begin{aligned} P &= \frac{10000}{(1.03)^{15}} + \frac{1000}{0.03}\left(1 - (1.03)^{-15}\right) \\ &= 10000 \cdot 0.64 + \frac{1000}{0.03}(1 - 0.64) \\ &= 6400 + 1000\frac{.36}{.03} = 6400 + 1000 \cdot 12 \\ &= 18400. \end{aligned}$$

That is, after 15 years the bond is now worth almost twice what it was purchased for.

2. *Linear Programming*

(a)(40 points) A municipality has the following schedule of liabilities:

Year	2005	2006	2007	2008	2009	2010	2011	2012
Dollars	12,000	18,000	20,000	20,000	16,000	15,000	12,000	10,000

The bonds available for purchase today are given in the following table. All bonds have a face value of \$100. The coupon figure is annual. For example, bond 5 costs \$98 today and pays back \$4 in 2005, \$4 in 2006, \$4 in 2007, and \$104 in 2008. All of these bonds are widely available and can be purchased in large quantities at the stated prices.

Bond	1	2	3	4	5	6	7	8	9	10
Price	102	99	101	98	98	104	100	101	102	104
Coupon	5%	4%	5%	4%	4%	5%	3%	4%	5%	5%
Maturity	2004	2005	2006	2007	2008	2009	2009	2010	2011	2012

Formulate a linear program to find the least cost portfolio of bonds to purchase today to meet the obligations of the municipality over the next 8 years. Assume that any surplus cash in a given year can be re-invested for a year at annual rate of 2.5%.

SOLUTION: The decision variables are $b_i,\ i = 1, \ldots, 10$ where

$$b_i = \textit{the number of bonds } \mathbf{i} \textit{ to purchase at the beginning of year 1,}$$

and $x_j,\ j = 1, \ldots, 8$ where

$$x_j = \textit{the amount invested at 2.5\% at the beginning of year } j.$$

The objective is to minimize the total ammount of money required to match the required cash flow stream.

$$\min x_1 + c^T b, \quad \text{where } c = (102, 99, 101, 98, 98, 104, 100, 101, 102, 104)^T .$$

The constraints require us to meet our obligations:

$$Ab + 1.025x \geq Kx + r,$$

where $r = 1000 \cdot (12, 18, 20, 20, 16, 15, 12, 10)^T$,

$$A = \begin{bmatrix} 105 & 4 & 5 & 4 & 4 & 5 & 3 & 4 & 5 & 5 \\ 0 & 104 & 105 & 4 & 4 & 5 & 3 & 4 & 5 & 5 \\ 0 & 0 & 0 & 104 & 4 & 5 & 3 & 4 & 5 & 5 \\ 0 & 0 & 0 & 0 & 104 & 5 & 3 & 4 & 5 & 5 \\ 0 & 0 & 0 & 0 & 0 & 105 & 103 & 4 & 5 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 104 & 5 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 105 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 105 \end{bmatrix} \text{ and } K = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Finally, all variables must be non-negative.

2.(b)(20 points) Let $A \in I\!R^{m \times n}$ and $c \in I\!R^m$. Use the Strong Duality Theorem of linear programming to show that the system

$$(I) \qquad Ax \leq 0 \quad \text{and} \quad c^T x > 0 \quad \text{is unsolvable}$$

if and only if the system

$$(II) \qquad A^T y = c \quad \text{and} \quad 0 \leq y \quad \text{is solvable.}$$

SOLUTION: Consider the LP

$$\begin{array}{lll} \mathcal{P} & \text{maximize} & c^T x \\ & \text{subject to} & Ax \leq 0 \end{array}$$

The dual of $\mathcal{P}$ is the LP

$$\begin{array}{lll} \mathcal{D} & \text{minimize} & 0^T y \\ & \text{subject to} & A^T y = c \end{array}$$

Note that $\mathcal{P}$ is always feasible with optimal value less than or equal to 0 since $x = 0$ is feasible for $\mathcal{P}$. We begin by showing that $(I) \Rightarrow (II)$. For this note that if x is feasible for $\mathcal{P}$, then so is λx for all $\lambda \geq 0$. Hence, if there is an x with $Ax \leq 0$ and $c^T x > 0$, then the optimal value in $\mathcal{P}$ is $+\infty$. Therefore, if statement (I) holds, then the optimal value in $\mathcal{P}$ is zero. Hence by the Strong Duality Theorem of linear programming $\mathcal{D}$ is feasible, or equivalently, statement (II) holds.

We now show that $(II) \Rightarrow (I)$. If statement (II) holds, then again by the Strong Duality Theorem of linear programming $\mathcal{P}$ is feasible, has optimal value zero, and this optimal value is attained. But then $c^T x \leq 0$ for all x satisfying $Ax \leq 0$, or equivalently, statement (I) holds.

3. *Nonlinear Programming*

Let $\ell \in I\!R^n$ and $u \in I\!R^n$ and consider the *box* $B = \{x \in I\!R^n \,:\, \ell_i \le x_i \le u_i, \text{ for } i = 1, 2, \dots, n\}$.

a)(30 points) Show that the vector $d \in I\!R^n$ is an element of the tangent cone to B at the point $x \in B$ if and only if

$$\begin{aligned} d_i &\ge 0 \text{ if } x_i = \ell_i, & (1)\\ d_i &\le 0 \text{ if } x_i = u_i, \text{ and} & (2)\\ d_i &\quad \text{is free to be any real number if } l_i < x_i < u_i\,. & (3)\end{aligned}$$

SOLUTION: We first show that any direction d satidfying (1)-(3) is a feasible direct, and so these directions must be contained in the tangent cone. To see this set

$$\bar{t} = \min\{|u_i - \ell_i|, x_j - \ell_j, u_j - x_j \mid i \in A(x),\ j \in N(x)\} > 0,$$

where $A(x) = A_\ell(x) \cup A_u(x)$,

$$\begin{aligned} A_\ell(x) &= \{i \mid x_i = \ell_i\}\\ A_u(x) &= \{i \mid x_i = u_i\}\end{aligned}$$

and

$$N(x) = \{j \mid \ell_i < x_i < u_i\}\,.$$

Then $x + td \in B$ for all $t \in [0, \bar{t}]$.

Next take $d \in T_B(x)$ so that there exists sequences $\{x^\nu\} \subset B$ and $t_\nu \downarrow 0$ such that $x^\nu \to x$ and $t_\nu^{-1}(x^\nu - x) = d^\nu \to d$. Now if $x_i = \ell_i$, then $0 \le t_\nu^{-1}(x_i^\nu - \ell_i) = d_i^\nu$ for all $\nu = 1, 2, \dots$ so that $d_i \ge 0$. If $x_i = u_i$, then $0 \ge t_\nu^{-1}(x_i^\nu - u_i) = d_i^\nu$ for all $\nu = 1, 2, \dots$ so that $d_i \le 0$. Hence d satisfies (1)-(3), which proves the result.

b)(30 points) Let $f : I\!R^n \to I\!R$ be continuously differentiable. Show that if $\bar{x} \in B$ solves the problem $\min\{f(x) \,:\, x \in B\}$, then

$$\begin{aligned} (\nabla f(\bar{x}))_i &\ge 0 \quad \text{if } x_i = \ell_i,\\ (\nabla f(\bar{x}))_i &\le 0 \quad \text{if } x_i = u_i,\\ (\nabla f(\bar{x}))_i &= 0 \quad \text{otherwise.}\end{aligned}$$

SOLUTION: The KKT conditions for the problem $\min\{f(x) \,:\, x \in B\}$ are

$$\begin{aligned} \ell \le x \le u && (4)\\ 0 \le w,\ 0 \le z && (5)\\ 0 = w^T(x - \ell),\ 0 = z^T(u - x) && (6)\\ 0 = \nabla f(x) - w + z\,. && (7)\end{aligned}$$

The complementarity conditions (6) plus the primal and dual feasibility conditions (4) and (5) imply that

$$\begin{aligned} w_i = 0 &= z_i \quad \text{for} \quad i \in N(x), & (8)\\ z_i = 0 &\le w_i \quad \text{for} \quad i \in A_\ell(x) \text{ and} & (9)\\ w_i = 0 &\le w_i \quad \text{for} \quad i \in A_u(x). & (10)\end{aligned}$$

By combining the stationarity condition (7) with (8)-(10) we get

$$\begin{aligned} (\nabla f(x))_i &= 0 \quad \text{for} \quad i \in N(x),\\ (\nabla f(x))_i &\ge 0 \quad \text{for} \quad i \in A_\ell(x) \text{ and}\\ (\nabla f(x))_i &\le 0 \quad \text{for} \quad i \in A_u(x)\,,\end{aligned}$$

which proves the result.

4. *Convexity*

(a) (30 points) Let $A \in \mathbb{R}^{m\times n}$ and $C \subset \mathbb{R}^m$. Show that the set

$$\{x \mid Ax \in C, x \in \mathbb{R}^n\}$$

is a convex subset of $\mathbb{R}^n$.

SOLUTION: Let $x, y \in \{x \mid Ax \in C, x \in \mathbb{R}^n\}$ and $0 \le \lambda \le 1$. Then

$$A\left((1-\lambda)x + \lambda y\right) = (1-\lambda)Ax + \lambda Ay \in C$$

since C is a convex set. Hence $(1-\lambda)x + \lambda y \in \{x \mid Ax \in C, x \in \mathbb{R}^n\}$, and so this set must be convex.

(b) (30 points) Let $\gamma \in \mathbb{R}$, $g \in \mathbb{R}^n$, and $H \in \mathbb{R}^{n\times n}$ with H symmetric, and consider the quadratic function

$$q(x) = \gamma + g^T x + \frac{1}{2}x^T H x\ .$$

Given $\bar{x} \in \mathbb{R}^n$ show that there always exists $\alpha \ge 0$ such that the function

$$g_\alpha(x) = g(x) + \frac{\alpha}{2}\|x - \bar{x}\|^2$$

is convex.

BONUS POINTS: (5 points) What is the smallest value that α can take that insures that g_α is convex?

SOLUTION: Note that $\nabla^2 g_\alpha(x) = H + \alpha I$. If λ is and eigenvector of H with associated eigenvector x, then $(H+\alpha I)x = Hx + \alpha x = (\lambda+\alpha)x$ so that x is an eigenvector of $H+\alpha I$ with associated eigenvalue $\lambda + \alpha$. This implies that the eigenvalues of $H + \alpha I$ are $\lambda_i + \alpha,\ i = 1,\ldots,n$, where $\lambda_i,\ i = 1,\ldots,n$ are the eigenvalues of H. Hence if we choose

$$\alpha \ge \max\left\{-\min\{0, \lambda_i\} \mid i = 1,\ldots,n\right\},$$

then $\lambda_i + \alpha \ge 0,\ i = 1,\ldots,n$, which implies that the matrix $\nabla^2 g_\alpha(x) = H + \alpha I$ is positive semi-definite, which, in turn, implies that the function g_α is convex.

This argument shows that the smallest α can be chosen is $\alpha = \max\left\{-\min\{0, \lambda_i\} \mid i = 1,\ldots,n\right\}$.

5. *The Markowitz QP*

(a) (20 points) Consider the *no short selling* minimum variance Markowitz portfolio problem

$$\mathcal{M}_{ns} \; : \; \begin{array}{ll} \text{minimize} & \frac{1}{2} w^T \Sigma w \\ \text{subject to} & 0 \leq w, \; e^T w = 1 \; , \end{array}$$

where $\Sigma \in I\!R^{n \times n}$ is symmetric and positive definite, and $e \in I\!R^n$ is the vector of all ones. Use the complementarity conditions for this problem to show that the unique solution must satisfy

$$e \leq \frac{\Sigma w}{w^T \Sigma w}.$$

SOLUTION: The KKT condition for $\mathcal{M}$ are

$$\begin{aligned} 0 \leq w, \; e^T w = 1 \\ 0 \leq z \\ z^T w = 0 \\ 0 = \Sigma w - \gamma e - z \; . \end{aligned}$$

By eliminating z, these reduce to

$$\begin{aligned} 0 \leq w, \; e^T w = 1 \\ w^T (\Sigma w - \gamma e) = 0 \\ \Sigma w - \gamma e \geq 0 \; . \end{aligned}$$

The second of these equations implies that $\gamma = w^T \Sigma w > 0$ where the strict inequality follow from the fact that Σ is positive definite and $w \neq 0$ since $e^T w = 1$. Plugging $\gamma = w^T \Sigma w$ into the inequality $\Sigma w - \gamma e \geq 0$ and dividiing through by $w^T \Sigma w$ gives the desired result.

5.(b)(40 points) Consider the following Markowitz mean-variance portfolio optimization problem for the two financial assets $i = 1, 2$ having rates of return r_1 and r_2:

$$E(r_1) = 0.02, \ E(r_2) = 0.01, \ E(r_3) = 0.02,$$
$$\text{var}(r_1) = 0.04, \ \text{var}(r_2) = 0.02, \ \text{var}(r_3) = 0.02$$
$$\text{cov}(r_1, r_2) = 0.02, \ \text{cov}(r_1, r_3) = 0, \ \text{and} \ \text{cov}(r_2, r_3) = 0.$$

The target rate of return on the portfolio is 0.03.

(i) What is the covariance matrix Σ for r_1 and r_2 and what is its inverse?

SOLUTION:

$$\Sigma = \frac{1}{100}\begin{bmatrix} 4 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \text{and} \quad \Sigma^{-1} = 25\begin{bmatrix} 2 & -2 & 0 \\ -2 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix} .$$

(ii) Solve this Markowitz QP.

SOLUTION: Clearly the problem is feasible since $\mathrm{m} = (1/100)(2\ 1\ 2)^{\mathrm{T}}$ is not parallel to e. Since

$$\Sigma^{-1}\mathrm{e} = 25(0\ 2\ 2)^{\mathrm{T}}$$

the minimum variance solution is

$$w_{mv} = (1/2)(0\ 1\ 1)^T .$$

Since $\mathrm{m}^{\mathrm{T}}\mathrm{w}_{\mathrm{mv}} = 0.015 < 0.03$, w_{mv} does not solve $\mathcal{M}$. Since

$$\Sigma^{-1}\mathrm{m} = (1/2\ 0\ 1)^{\mathrm{T}}$$

the market weights are

$$w_{mk} = (1/3)(1\ 0\ 2)^T.$$

The solution then has the form

$$w = w_{mv} + \alpha(w_{mk} - w_{mv}) = \frac{1}{2}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \frac{\alpha}{6}\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} .$$

Since $0.03 = \mathrm{m}^{\mathrm{T}}\mathrm{w}$, we have $\alpha = 3$ so

$$w = \frac{1}{2}\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} .$$