The midterm consist of two parts and each part will have 3 multipart questions. Each of the 6 questions is worth 50 points for a total of 300 points. The two parts of the exam are (I) Linear Least Squares and (II) Quadratic Optimization. In each part, the first question concerns definitions, theorems, and proofs, and the remaining two questions are computational. This format is very much like the quizzes. More detailed descriptions of the questions are given below. This is then followed by a list of sample questions.

I Linear Least Squares

1 Theory Question: For this question you will need to review all of the vocabulary words as well as the theorems for Linear Least Squares. You will also be asked to establish properties of the linear algebraic, structures such as orthogonal projections, and the solution set.

2 Linear Algebra: Here you may be asked to compute the solution to an associated linear system, such as the normal equations, compute an LU and/or QR factorization, compute an orthogonal projection onto a subspace, and /or compute a representation for the solution for a specially structured linear least squares problem. In addition, you may be asked to use the LU or QR factorizations to solve the normal equations. Specifically, you amy be asked to apply the *recipe* for using the QR factorization to solve the normal equations.

3 Other Computations: Here you may be asked to compute the solution set to a linear least squares problem, compute and orthogonal projection onto a subspace, or find the least 2-norm solution to a consistent linear system.

II Quadratic Optimization

4 Theory Question: For this question you will need to review all of the vocabulary words as well as the theorems in Sections 1-3 of Chapter 3. You will also be asked to establish properties of the linear algebraic structures (symmetric matrices in particular) and the solution sets to unconstrained and affinely constrained quadratic optimization problems.

5 Linear Algebra: Here you may be asked to compute a Cholesky factorization, and/or a representation for the solution for a specially structured quadratic optimization problem.

6 Other Computations: Here you may be asked to compute the solution set to a quadratic optimization problem with or without affine constraints.

Sample Questions

More sample problems will be added over the weekend. Solutions will be posted on Sunday.

(I) Linear Least Squares

Question 1:

Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$, and consider the linear least squares problem

$$\mathcal{LLS} \qquad \min \frac{1}{2}\|Ax-b\|_2^2 \ .$$

a. Show that the matrix A^TA is always positive semi-definite and provide necessary and sufficient condition on A under which A^TA is positive definite.

b. Show that $\mathrm{Nul}(A^TA) = \mathrm{Nul}(A)$ and $\mathrm{Ran}(A^TA) = \mathrm{Ran}(A^T)$.

c. Show that $\mathcal{LLS}$ always has a solution.

d. State and prove a necessary and sufficient condition on the matrix $A \in \mathbb{R}^{m\times n}$ under which $\mathcal{LLS}$ has a unique global optimal solution.

e. Decribe the QR-factorization of A and show how it can be used to construct a solution to $\mathcal{LLS}$.

f. How can the QR factorization of A be used to obtain the orthoronal projection onto $\text{Ran}(A)$.

g. If $\text{Nul}(A) = \{0\}$, show that $(A^TA)^{-1}$ is well defined and that $P = A(A^TA)^{-1}A^T$ is the orthogonal projection onto $\text{Ran}(A)$ with

$$\frac{1}{2}\|(I-P)b\|_2^2 = \min \frac{1}{2}\|Ax-b\|_2^2 \ .$$

h. Let $A \in \mathbb{R}^{m\times n}$ be such that $\text{Ran}(A) = \mathbb{R}^m$ and set $P = A^T(AA^T)^{-1}A$. Show that $Px^0 = A^T(AA^T)^{-1}b$ for every $x^0 \in \mathbb{R}^n$ satisfying $Ax^0 = b$ and that $\hat{x} := A^T(AA^T)^{-1}b$ is the unique solution to the problem

$$\min \frac{1}{2}\|x\|_2^2 \quad \text{subject to} \quad Ax = b \ .$$

Question 2:

(A) Consider the matrix and vector

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} .$$

a. Compute the orthogonal projection onto $\text{Ran}(A)$.
b. Compute the orthogonal projection onto $\text{Null}(A^T)$.
c. Compute the QR factorization of A.
d. Compute the LU factorization of A^TA.
e. Compute the solution for the LLS problem for this matrix and vector using the LU factorization of A^TA.
f. Compute the solution for the LLS problem for this matrix and vector using the QR factorization of A.

(B) Consider the matrix and vector

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 2 \\ 1 & -1 & 0 \\ 1 & 1 & 2 \end{bmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} .$$

a. Compute the orthogonal projection onto $\text{Ran}(A)$.
b. Compute the orthogonal projection onto $\text{Null}(A^T)$.
c. Compute the QR factorization of A.
d. Compute the LU factorization of A^TA.
e. Compute the solution for the LLS problem for this matrix and vector using the LU factorization of A^TA.

f. Compute the solution for the LLS problem for this matrix and vector using the QR factorization of A.

Question 3:

(A) Let $a \in \mathbb{R}$ and consider the function

$$f(x_1, x_2, x_3) = \frac{1}{2}[x_1^2 + (x_1 - 2a^4)^2 + (x_1 - x_2)^2 + (ax_2 + x_3)^2].$$

(a) Write this function in the form of the objective function for a linear least squares problem by specifying the matrix A and the vector b.

(b) Describe the solution set of this linear least squares problem as a function of a.

(B) Find the quadratic polynomial $p(t) = x_0 + x_1 t + x_2 t^2$ that best fits the following data in the least-squares sense:

t	-2	-1	0	1	2
y	2	-10	0	2	1

.

(C) Consider the LLS problem with

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 2 \\ 1 & -1 & 0 \\ 1 & 1 & 2 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

(a) What are the normal equations for this A and b.

(b) Solve the normal equations to obtain a solution to the problem LLS for this A and b.

(c) What is the general reduced QR factorization for the matrix A?

(d) Compute the orthogonal projection onto the range of A.

(e) Use the recipe given in the notes for solving the LLS problem associated with this A and b using the QR-factorization you have computed for A.

(f) If $\bar{x}$ solves LLS for this A and b, what is $A\bar{x} - b$?

(II) Quadratic Optimization

Question 4:

Consider the function

$$f(x) = \frac{1}{2}x^T Q x + c^T x,$$

where $Q \in \mathbb{R}^{n \times n}$ is symmetric and $c \in \mathbb{R}^m$.

(a) Give necessary and sufficient conditions on Q and c for which there exists a solution to the problem $\min_{x \in \mathbb{R}^n} f(x)$.

(b) If Q is positive definite, show that there is a nonsingular matrix L such that $Q = LL^T$.

(c) With Q and L as defined in the part (2), show that

$$f(x) = \frac{1}{2}\|L^T x + L^{-1}c\|_2^2 - \frac{1}{2}c^T Q^{-1} c.$$

(d) If Q is psd, under what conditions is $\min_{x \in \mathbb{R}^n} f(x) = -\infty$?

(e) Let $\hat{x} \in \mathbb{R}^n$ and S be a subspace of $\mathbb{R}^n$. Give necessary and sufficient conditions on Q and c for which there exists a solution to the problem

$$\min_{x \in \hat{x}+S} f(x) \ .$$

(f) Show that every local solution to the problem $\min_{x\in\mathbb{R}^n} f(x)$ is necessarily a global solution.

(g) Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$ and consider the problem $\min \left\{ \frac{1}{2}x^T Qx + c^T x \mid Ax = b \right\}$, where it is assumed that the system $Ax = b$ is consistent.

(i) Using a Lagrange multiplier vector $y \in \mathbb{R}^m$ give a necessary and sufficient condition under which $\bar{x} \in \mathbb{R}^m$ is and optimal solution to this problem.

(ii) Under what conditions is $(\bar{x}, \bar{y})$ the unique solution and Lagrange multipier pair for this problem.

(iii) Provide a necessary and sufficient condition under which $\bar{x}$ is a unique solution to this problem.

Question 5:

(A) Compute the Cholesky factorizations of the following matrices.

$$(a)\ H = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \qquad (b)\ H = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

$$(c)\ H = \begin{bmatrix} 5 & 2 & -1 \\ 2 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix} \qquad (d)\ H = \begin{bmatrix} 1 & 4 & 1 \\ 4 & 20 & 2 \\ 1 & 2 & 2 \end{bmatrix} .$$

Question 6:

(A) For each of the matrices H and vectors g below, determine the optimal value in

$$\mathcal{Q} : \min_x f(x) := \frac{1}{2}x^T Hx + g^T x.$$

If an optimal solution exists, compute the complete set of optimal solutions.

a.

$$H = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \quad \text{and} \quad g = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} .$$

b.

$$H = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & -2 \end{bmatrix} \quad \text{and} \quad g = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} .$$

c.

$$H = \begin{bmatrix} 5 & 2 & -1 \\ 2 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix} \quad \text{and} \quad g = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} .$$

(B) Consider the matrix $H \in \mathbb{R}^{3\times 3}$ and vector $g \in \mathbb{R}^3$ given by

$$H = \begin{bmatrix} 1 & 4 & 1 \\ 4 & 20 & 2 \\ 1 & 2 & 2 \end{bmatrix} \quad \text{and} \quad g = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} .$$

Does there exists a vector $u \in \mathbb{R}^3$ such that $f(tu) \xrightarrow{t\uparrow\infty} -\infty$? If yes, construct u.

(C) Consider the linearly constrained quadratic optimization problem

$$\mathcal{Q}(H,g,A,b) \qquad \begin{array}{ll} \text{minimize} & \frac{1}{2}x^T Hx + g^T x \\ \text{subject to} & Ax = b \ , \end{array}$$

where H, A, g, and b are given by

$$H = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}, \ g = (1,1,1)^T, \ b = (4,2)^T, \quad \text{and} \quad A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

a. Compute a basis for the null space of A.

b. Solve the problem $\mathcal{Q}(H,g,A,b)$ two different ways. One should use a basis for the null space of A and the other should not.

(D) Let $H \in \mathbb{R}^{n\times n}$ be symmetric and positive definite, $r \in \mathbb{R}^n \setminus \text{Span}[\mathsf{e}]$, $\mu \in \mathbb{R}$. Solve the problem

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}x^T Hx \\ \text{subject to} & \mathsf{e}^T x = 1 \quad \text{and} \quad r^T x = \mu \ , \end{array}$$

where $\mathsf{e} := (1,1,\ldots,1)^T \in \mathbb{R}^n$ is the vector of all ones and $\mu \in \mathbb{R}$.