Portfolio Modeling Using LPs

Recall that a linear program (LP) is an optimization problem wherein one minimizes of maximizes a linear objective function in a finite number of variables subject to a finite number of linear inequality and/or equality constraints. The variables in such a problem are typically refered to as the *decision variables*. The basic modeling paradigm for linear programming is the following.

LP Modeling Paradigm

Step 1: Determine the decision variables and lable them. The decision variables are those variables whose values must be determined in order to execute a plan of action or strategy regardless of the optimality of this plan.

Step 2: Determine the objective and write an expression for it that is linear in the decision variables.

Step 3: Determine the explicit (or stated) constraints and write them as either equality or inequality constraints that are linear in the decision variables.

Step 4: Determine the implicit constraints. These are the constraints that are not explicitly stated but are required in order for the problem to be physically meaningful. These constraints must also be written as either equality or inequality constraints that are linear in the decision variables.

The hardest part of modeling an LP is the determination of the decision variables. Once this is done usually the remainder of the modeling follows in a reasonably straightforward way. In determining the decision variables put yourself in the shoes of the person that must execute the plan or strategy to be implemented. This is not the person that determines the plan or strategy, rather it is the person that exercises it, and as such does not need to know or even care that it is in some way optimal. Ask yourself "what is the practical hands on information that this person absolutely must have in order to do their job". It is this information that determines the decision variables.

Perhaps the most common beginners mistake in modeling a problem as an LP (or, modeling an optimization problem in general) is to try to solve the problem as they model it. This attempt to solve the problem as it is modeled takes many forms such as determining that an inequality constraint will really be an equality constraint at the solution, or certain variables can be written in terms of others, or some decision variables will attain a certain value at solution for *obvious* reasons, etc. ... NEVER ATTEMPT TO SOLVE THE PROBLEM IN THE MODELING STAGE!!! Also, use as many variables as you need to make sense of the problem. Don't worry that there may be a lot of them. It's not your problem since you are not trying to solve the problem only model it.

We now illustrate this modeling paradigm with a simple portfolio optimization problem.

Portfolio Optimization with LPs

Suppose that you have $20,000 to invest and you have chosen four different financial instruments for structuring your investment. These are as follows:

1. Buy stock X which is currently selling for $20 per share.
2. Purchase *European call* options to buy a share of stock X at $15 in exactly 6 months time. These options are selling today for $10 each.
3. Raise more funds for investment immediately by selling the European call options described above under the same terms.
4. Purchase 6 month riskless zero-coupon bonds having a face value of $100 at a cost of $90 each today.

Remark (**European Call Options**)
A European call option to buy a stock at a price x and *exercise date* d gives the purchaser the right to buy the stock at x dollars on precisely the date d. The purchaser of the option need not exercise this right to buy, and they can only exercise this right on the date d (not before or after).

You determine that there are three equaly likely scenarios that may occur to the stock price for stock X:

Scenario 1: Stock X sells for $20 a share in 6 months.

Scenario 2: Stock X sells for \$40 a share in 6 months.

Scenario 2: Stock X sells for \$12 a share in 6 months.

Due to the heavy risks involved with selling options, the exchange has placed a margin requirement on the total number of European calls on stock X that you can sell. You may only write 5000 such calls. Formulate an LP to determine the portfolio of stocks, bonds, and options that maximizes your expected profit in 6 months.

Determine the Decision Variables.

$$\begin{aligned} S &= \text{the number of stock X purchased} \\ B &= \text{the number of bonds purchased} \\ C_1 &= \text{the number of call options purchased} \\ C_2 &= \text{the number of call options sold} \end{aligned}$$

Determine the Objective

The objective is to maximize the expected profit from the portfolio. To determine this we must determine the expected profit from each of the investment alternatives.

Expected Profit

Bonds: \$10 per bond

Stock: The profit on the stock depends on the three equally likely scenarios. In scenario 1 the profit is 0, and this happens with probability 1/3. In scenario 2 the profit is 20, and this happens with probability 1/3. In scenario 3 the profit is -8, and this happens with probability 1/3. Hence the expected profit on one stock of X is

$$\frac{1}{3}(0) + \frac{1}{3}(20) + \frac{1}{3}(-8) = 4.$$

Call Options Bought: Again, the profit on the call options purchases depends on the three equally likely scenarios. In scenario 1 we exercise the option and buy the stock for \$15, then turn around and sell it again for \$20 yielding a profit of \$5. We then subtract from this \$5 the initial cost of the option (\$10) to obtain an overall loss of \$5, or equivalently, a profit of -\$5. In scenario 2 we exercise the option and buy the stock for \$15, then turn around and sell it again for \$40 yielding a profit of \$25. We then subtract from this \$25 the initial cost of the option (\$10) to obtain an overall profit of \$15. Finally, in scenario 3 we do not exercise the option to buy the stock at \$15 since it now selling for less than that. Hence our loss in this case equals the purchase price of the option, or equivalently, we obtain a profit on -\$10 under this scenario. Since each of these scenarios is equally likely the expected profit from the purchase of a call option is

$$\frac{1}{3}(-5) + \frac{1}{3}(15) + \frac{1}{3}(-10) = 0.$$

Call Options Sold: As we have seen, the profit on the call options purchases depends on the three equally likely scenarios. In scenario 1 the purchaser of the option to buy exercises this option. In this case, we must buy the stock for \$20 and then turn around and sell it to the purchaser of the option for \$15, thus incuring a loss of \$5. However, the purchaser initially paid us \$10 for the option. Hence our overall profit under this scenario is \$5. In scenario 2 the purchaser of the option to buy again exercises this option. In this case, we must buy the stock for \$40 and then turn around and sell it to the purchaser of the option for \$15, thus incuring a loss of \$25. However, the purchaser initially paid us \$10 for the option. Hence our overall profit under this scenario is -\$15. In scenario 3 the purchaser of the option does not exercise the option and so our profit in this scenario is the \$10 the purchaser initially gave us for the option. Since each of these scenarios is equally likely the expected profit from the sale of a call option is

$$\frac{1}{3}(5) + \frac{1}{3}(-15) + \frac{1}{3}(10) = 0.$$

Therefore, the objective in this problem is to maximize the linear expression $10B + 4S$.

Determine the Explicit Constraints

Budget Constraint: $90B + 20S + 10C_1 \leq 20000 + 10C_2$

Margin Constraint: $C_2 \leq 5000$

Determine the Implicit Constraints

All of the decision variables must be positive:

$$0 \le S,\ 0 \le B,\ 0 \le C_1,\ \text{and } 0 \le C_2.$$

The complete LP tkes the form

$$\begin{array}{ll} \text{maximize} & 10B \ + \ 4S \\ \text{subject to} & 90B + 20S + 10(C_1 - C_2) \ \le \ 20000 \\ & 0 \le S,\ 0 \le B,\ 0 \le C_1,\ \text{and } 0 \le C_2 \le 5000. \end{array}$$

There is one simplification in this problem that is very convenient and will be employed in future problems of this type. This involves the variables C_1 and C_2. Observe that if $C_2 < 5000$ at the solution and N is any number between C_2 and 5000, then we may change C_2 to $C_2 + N$ and C_1 to $C_1 + N$ and obtain a new solution that must also be optimal. The effect here is to buy N calls and turn around and immediately sell them again. This is of course wasteful activity that is not dis-allowed in our modeling. To avoid this we introduce a new variable $C = C_1 - C_2$. Negative values of C correspond to selling calls, while positive values of C corresponds to buying them. The new and equivalent LP is

$$\begin{array}{ll} \text{maximize} & 10B \ + \ 4S \\ \text{subject to} & 90B + 20S + 10C \ \le \ 20000 \\ & 0 \le S,\ 0 \le B,\ -5000 \le C. \end{array}$$

The solution of this LP gives

$$\begin{aligned} B &= 0 \\ S &= 3500 \\ C &= -5000, \end{aligned}$$

with optimal value giving an expected profit of \$14000. But remember that this is only the *expected* profit. In the event that scenario 2 actually occurs, we actually incur a loss of \$5000 since

$$20S + 15C = 20 \times 3500 - 15 \times 5000 = -5000.$$

As an alternative, one might choose to add the condition that the profit from the portfolio must be at least \$2000. This can be done by adding further constraints that require that regardless of the scenario the profit must exceed \$2000. To do this, we simply lower bound the revenue at the end of 6 month by \$22000. These lower bounding constraints are

Scenario 1: $100B + 20S + 5C \geq 22000$

Scenario 2: $100B + 40S + 25C \geq 22000$

Scenario 3: $100B + 12S \geq 22000$.

This gives a new LP of the form

$$\begin{array}{ll} \text{maximize} & 10B \ + \ 4S \\ \text{subject to} & 90B + 20S + 10C \leq 20000 \\ & 100B + 20S + 5C \geq 22000 \\ & 100B + 40S + 25C \geq 22000 \\ & 100B + 12S \geq 22000 \\ & 0 \leq S, \ 0 \leq B, \ -5000 \leq C. \end{array}$$

This LP yields the solution

$$\begin{array}{rcl} B & = & 0 \\ S & = & 2800 \\ C & = & -3600, \end{array}$$

with optimal value giving an expected profit of \$11200. But again it must be remembered that this is only the *expected* profit. Indeed, if scenario 2 occurs, then the actual profit will be \$2000 which is the least amount required.

To overcome the potential downside of any given scenario, we may choose instead to maximize the smallest possible profit regardless of which scenario occurs. We can do this by introducing a new variable Z representing the least possible revenue that we may have at the end of 6 months, regarless of which scenario occurs. We then maximize Z. This gives rise to the new LP

$$\begin{array}{ll} \text{maximize} & Z \\ \text{subject to} & 90B + 20S + 10C \leq 20000 \\ & 100B + 20S + 5C \geq 20000 + Z \\ & 100B + 40S + 25C \geq 20000 + Z \\ & 100B + 12S \geq 20000 + Z \\ & 0 \leq S, \ 0 \leq B, \ -5000 \leq C. \end{array}$$

The solution of this LP is

$$\begin{array}{rcl} B & = & 0 \\ S & = & 2273 \\ C & = & -2545, \end{array}$$

giving an expected profit of \$9092. The least possible profit from this solution is \$7285.