Markowitz Mean-Variance Portfolio Theory

1. Portfolio Return Rates

An investment instrument that can be bought and sold is often called an *asset*. Suppose we purchase an asset for x_0 dollars on one date and then later sell it for x_1 dollars. We call the ratio

$$R = \frac{x_1}{x_0}$$

the *return* on the asset. The *rate of return* on the asset is given by

$$r = \frac{x_1 - x_0}{x_0} = R - 1.$$

Therefore,

$$x_1 = Rx_0 \quad \text{and} \quad x_1 = (1+r)x_0 \ .$$

Sometimes it is possible to sell and asset that we do not own. This is called *short selling*. It works somewhat as follows. Suppose you wish to short (or short sell) a particular stock XXX. You begin by asking your stock broker if their firm is holding XXX in the total pool of stocks owned by all of their customers. If the brokerage does hold (or manage) some of stock XXX, you can ask them sell any number of stock XXX up to the number that they hold. This sale is credited against your account as a debt equal to the number of stock XXX they sell on your behalf. That is, your debt is not denominated in dollars, but rather in the number of stock XXX that you are shorting (i.e. your account is *short* by the given number of stock XXX). On your account asset sheet, this short sale appears as a negative number associated with the shorted asset. Remember, this negative number is not denominated in dollars, but rather in the number of stocks , or assets, shorted. Due to the sale of stock XXX you have received x_0 dollars. Eventually, you must ask the brokerage to buy the same number of stock XXX back as you originally asked them to sell and return this stock to the pool of assets that they are holding for their customers. On the date at which you return stock XXX you ask your broker to re-purchase it at its current going value of x_1 dollars and return it to the brokerage's asset pool. If $x_1 < x_0$, then you have made a profit on this transaction; otherwise, you have a loss. The return and rate of return on this transaction are given by

$$R = \frac{-x_1}{-x_0} = \frac{x_1}{x_0} \quad \text{and} \quad r = \frac{(-x_1)-(-x_0)}{-x_0} = \frac{x_1 - x_0}{x_0},$$

respectively. Short selling can be very risky, and many brokerage firms do not allow it. Nonetheless, it can be profitable.

Let us now consider constructing a portfolio consisting of n assets. We have an initial budget of x_0 dollars that we wish to assign to these assets. The amount that we assign to asset i is $x_{0i} = w_i x_0$ for $i = 1, 2, \ldots, n$, where w_i is a weighting factor for asset i. We allow the weights to take negative values, and when negative it means that the asset is being shorted in our portfolio. To preserve the budget constraint we require that the weights sum to 1: $\sum_{i=1}^{n} w_i = 1$. That is,

$$\text{the sum of the investments} = \sum_{i=1}^{n} w_i x_0 = x_0 \sum_{i=1}^{n} w_i = x_0 \ .$$

Notice that by shorting some stocks we open up more funds for the purchase of other stocks, because when we short a stock we receive the dollar value of that stock today and we can turn around and re-invest those dollars elsewhere with the purchase of other assets.

If R_i denotes the return on asset i, then the total receipts from our portfolio is

$$x_1 = \sum_{i=1}^{n} R_i w_i x_0 = x_0 \sum_{i=1}^{n} R_i w_i,$$

and so the total return from the portfolio is

$$R = \sum_{i=1}^{n} R_i w_i.$$

In addition, we have that the rate of return from asset i is $r_i = R_i - 1$, $i = 1, 2, \ldots, n$. Hence the rate of return on the portfolio is

$$r = R - 1 = (\sum_{i=1}^{n} R_i w_i) - (\sum_{i=1}^{n} w_i) = \sum_{i=1}^{n} (R_i - 1) w_i = \sum_{i=1}^{n} r_i w_i \ .$$

2. The Basics of Markowitz Mean-Variance Portfolio Theory

In the Markowitz mean-variance portfolio theory, one models the rate of returns on assets as random variables. The goal is then to choose the portfolio weighting factors *optimally*. In the context of the Markowitz theory an optimal set of weights is one in which the portfolio achieves an acceptable baseline expected rate of return with minimal *volatility*. Here the variance of the rate of return of an instrument is taken as a surrogate for its volatility.

Let r_i be the random variable associated with the rate of return for asset i, for $i = 1, 2, \ldots, n$, and define the random vector

$$z = \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{pmatrix} .$$

Set $\mu_i = E(r_i)$, $\mathrm{m} = (\mu_1, \mu_2, \ldots, \mu_n)^T$, and $\mathrm{cov}(z) = \Sigma$. If $w = (w_1, w_2, \ldots, w_n)^T$ is a set of weights associated with a portfolio, then the rate of return of this portfolio $r = \sum_{i=1}^n r_i w_i$ is also a random variable with mean $\mathrm{m}^T w$ and variance $w^T \Sigma w$. If μ_b is the acceptable baseline expected rate of return, then in the Markowitz theory an optimal portfolio is any portfolio solving the following quadratic program:

$$\begin{array}{lll} \mathcal{M} & \text{minimize} & \frac{1}{2} w^T \Sigma w \\ & \text{subject to} & \mathrm{m}^T w \geq \mu_b, \text{ and } \mathrm{e}^T w = 1 \ , \end{array}$$

where e always denotes the vector of ones, i.e., each of the components of e is the number 1. The KKT conditions for this quadratic program are

$$0 = \Sigma w - \lambda \mathrm{m} - \gamma \mathrm{e} \tag{1}$$

$$\mu_b \leq \mathrm{m}^T w, \ \mathrm{e}^T w = 1, \ 0 \leq \lambda \tag{2}$$

$$\lambda^T (\mathrm{m}^T w - \mu_b) = 0 \tag{3}$$

for some $\lambda, \ \gamma \in I\!R$. Since the covariance matrix is symmetric and positive definite, we know that if (w, λ, γ) is any triple satisfying the KKT conditions then w is necessarily a solution to $\mathcal{M}$. Indeed, it is easily shown that if $\mathcal{M}$ is feasible, then a solution to $\mathcal{M}$ must always exist and so a KKT triple can always be found for $\mathcal{M}$.

Proposition 2.1. *Let $A \in I\!R^{m\times n}$, $B \in I\!R^{m\times t}$, $E \in I\!R^{s\times n}$, $F \in I\!R^{s\times t}$, $M \in I\!R^{t\times n}$, $Q \in I\!R^{n\times n}$ and $H \in I\!R^{t\times t}$ with Q and H symmetric, and let $r \in I\!R^m$, and $h \in I\!R^s$. Further assume that the symmetric matrix*

$$\hat{Q} = \begin{bmatrix} Q & M^T \\ M & H \end{bmatrix}$$

is positive semi-definite. If the following quadratic program is feasible, then it has finite optimal value and a solution attaining this optimal value exists:

$$\begin{array}{ll} \textit{minimize} & \frac{1}{2}\left[u^T Q u + 2 v^T M u + v^T H v\right] \\ \textit{subject to} & Au + Bv \leq r \\ & Eu + Fv = h \\ & 0 \leq u \ . \end{array}$$

Proof. Let S be any square root of the matrix $\hat{Q}$, and set $x = (u^T, v^T)^T$. Set $\Omega = I\!R^m_+ \times I\!R^s \times I\!R^n_+$. Then the constraint region for the QP can be written as $\mathcal{F} = \{x \in I\!R^n \times I\!R^t \,:\, Tx \in \Omega\}$, where

$$T = \begin{bmatrix} A & B \\ E & F \\ I & 0 \end{bmatrix} .$$

By assumption $\mathcal{F} \neq \emptyset$. Making the change of variable $y = Sx$ in the QP yields the QP

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}\|y\|^2 \\ \text{subject to} & y \in S\mathcal{F} = \{Sx \,:\, x \in \mathcal{F}\} . \end{array}$$

Since the set $\mathcal{F}$ is a closed nonempty polyhedral convex set, so is the set $S\mathcal{F}$. Hence the solution $\bar{y}$ to this QP is given by the point closest to the origin in the closed set $S\mathcal{F}$ which must exist since this set is closed and nonempty. Therefore, the solution set to the original QP is nonempty and is given by $\{x \,:\, \bar{y} = Sx,\ x \in \mathcal{F}\}$. $\square$

Assume that Σ is nonsingular and that $\bar{w}$ is a solution to $\mathcal{M}$ ($\bar{w}$ exists by Proposition 2.1). We consider two cases.

- $\mu_b < \mathrm{m}^T\bar{w}$: In this case, the complementarity condition (3) implies $\lambda = 0$. Hence the KKT conditions reduce to the two equations $0 = \Sigma\bar{w} - \gamma\mathrm{e}$ and $\mathrm{e}^T\bar{w} = 1$. Multiplying the first through by Σ^{-1} yields $\bar{w} = \gamma\Sigma^{-1}\mathrm{e}$. Multiplying this equation through by e and using the fact that $\mathrm{e}^T\bar{w} = 1$ gives $\gamma = (\mathrm{e}^T\Sigma^{-1}\mathrm{e})^{-1}$. Therefore,

  $$\bar{w} = (\mathrm{e}^T\Sigma^{-1}\mathrm{e})^{-1}\Sigma^{-1}\mathrm{e}.$$

  It is important to note that this value of w gives the smallest possible variance over all portfolios since it solves the problem

  $$\mathcal{M}_{\text{min-var}} : \begin{array}{ll} \text{minimize} & \frac{1}{2}w^T\Sigma w \\ \text{subject to} & \mathrm{e}^T w = 1 . \end{array}$$

  Consequently, the return associated with the least variance solution is

  $$\mu_{\text{min-var}} = \frac{m^T\Sigma^{-1}\mathrm{e}}{\mathrm{e}^T\Sigma^{-1}\mathrm{e}}.$$

  We denote the set of weights associated with the minimum variance solution $\bar{w}$ by $w_{\text{min-var}}$ as well.

  Finally observe that is the minimum variance weights $w_{\text{min-var}}$ are feasible for $\mathcal{M}$, that is, if $\mathrm{m}^T w_{\text{min-var}} \geq \mu_b$, then $w_{\text{min-var}}$ must be the solution to $\mathcal{M}$ since it the solution to the problem $\mathcal{M}_{\text{min-var}}$. Therefore, when solving $\mathcal{M}$ one first computes $w_{\text{min-var}}$ and checks to see if the inequality $\mathrm{m}^T w_{\text{min-var}} \geq \mu_b$

holds. If it does hold, then $w_{\text{min-var}}$ solves $\mathcal{M}$ and no further work is required. If it does not hold then you know that the constraint $\mathrm{m}^T w = \mu_b$ at the solution to $\mathcal{M}$.

- $\mu_b = \mathrm{m}^T \bar{w}$: Multiplying (1) through by Σ^{-1} gives

(4) $$\bar{w} = \lambda \Sigma^{-1}\mathrm{m} + \gamma \Sigma^{-1}\mathrm{e} \ .$$

Using this formula for $\bar{w}$ and (2), we get the two equations

$$\begin{aligned} \mu_b &= \lambda \mathrm{m}^T \Sigma^{-1}\mathrm{m} + \gamma \mathrm{m}^T \Sigma^{-1}\mathrm{e} \\ 1 &= \lambda \mathrm{m}^T \Sigma^{-1}\mathrm{e} + \gamma \mathrm{e}^T \Sigma^{-1}\mathrm{e} \ , \end{aligned}$$

or equivalently, the 2×2 matrix equation

(5) $$\begin{bmatrix} \mathrm{m}^T\Sigma^{-1}\mathrm{m} & \mathrm{m}^T\Sigma^{-1}\mathrm{e} \\ \mathrm{m}^T\Sigma^{-1}\mathrm{e} & \mathrm{e}^T\Sigma^{-1}\mathrm{e} \end{bmatrix} \begin{pmatrix} \lambda \\ \gamma \end{pmatrix} = \begin{pmatrix} \mu_b \\ 1 \end{pmatrix} .$$

Properties of positive definite matrices can be used to show that the matrix

$$T = \begin{bmatrix} \mathrm{m}^T\Sigma^{-1}\mathrm{m} & \mathrm{m}^T\Sigma^{-1}\mathrm{e} \\ \mathrm{m}^T\Sigma^{-1}\mathrm{e} & \mathrm{e}^T\Sigma^{-1}\mathrm{e} \end{bmatrix} = [\mathrm{m}\ \mathrm{e}]^T \Sigma^{-1} [\mathrm{m}\ \mathrm{e}]$$

is always positive semi-definite There are a number of ways to see this. The simplest is to exploit the factored form $T = [\mathrm{m}\ \mathrm{e}]^T \Sigma^{-1} [\mathrm{m}\ \mathrm{e}]$. But a simple test is to check that

$$0 < \delta = (\mathrm{m}^T\Sigma^{-1}\mathrm{m})(\mathrm{e}^T\Sigma^{-1}\mathrm{e}) - (\mathrm{m}^T\Sigma^{-1}\mathrm{e})^2.$$

This will always be the case whenever the vectors m and e are linearly independent.

If $\delta = 0$, then it must be the case that $m = \tau e$ for some $\tau \in I\!R$. In this case, if $\mu_b/\tau \neq 1$, then the problem $\mathcal{M}$ is necessarily infeasible. If $\mu_b/\tau = 1$, then $w_{\text{min-var}}$ solves $\mathcal{M}$ which would have been detected by first computing $w_{\text{min-var}}$ and then checking its feasibility for $\mathcal{M}$.

If $\delta > 0$, the system (5) can be solved to give

$$\lambda = \mathrm{e}^T v \quad \text{and} \quad \gamma = -\mathrm{m}^T v,$$

where

$$v = \delta^{-1}\Sigma^{-1}(\mu_b \mathrm{e} - \mathrm{m}) \ .$$

Plugging these values into (4) gives the optimal solution

$$\begin{aligned} w &= \frac{\Sigma^{-1}\mathrm{e}}{\mathrm{e}^T\Sigma^{-1}\mathrm{e}} + \alpha \left[\frac{\Sigma^{-1}\mathrm{m}}{\mathrm{e}^T\Sigma^{-1}m} - \frac{\Sigma^{-1}\mathrm{e}}{\mathrm{e}^T\Sigma^{-1}\mathrm{e}} \right] \\ &= (1-\alpha)\frac{\Sigma^{-1}\mathrm{e}}{\mathrm{e}^T\Sigma^{-1}\mathrm{e}} + \alpha \frac{\Sigma^{-1}\mathrm{m}}{\mathrm{e}^T\Sigma^{-1}m} \\ &= (1-\alpha) w_{\text{min-var}} + \alpha w_{\text{mk}} \ , \end{aligned}$$

where

$$w_{\text{mk}} = \frac{\Sigma^{-1}\mathrm{m}}{\mathrm{e}^T\Sigma^{-1}m}$$

and

$$\alpha = \frac{\mu_b(m^T\Sigma^{-1}e)(e^T\Sigma^{-1}e) - (m^T\Sigma^{-1}e)^2}{\delta} .$$

Observe that the optimal set of weights is a linear combination of the two sets of weights $w_{\text{min-var}}$ and w_{mk}, both of which satisfy the constraint $\mathrm{e}^T w = 1$. We have labeled weights w_{mk} as the *market* weights since they incorporate all of the market information on the assets under consideration.

Let us now recap the solution proceedure for $\mathcal{M}$.

SOLUTION PROCEEDURE

Check Feasibility

First check feasibility. For this we need only check to see if m is parallel to e. If it is, then $\mathrm{m} = \tau e$ for some $\tau \in I\!R$. In this case the problem is infeasible if $\mu_b > \tau$. If $\mathrm{m} = \tau e$ and $\mu_b \leq \tau$, then compute Σ^{-1} and evaluate the minimum variance weights $w_{\text{min-var}}$. These weights solve $\mathcal{M}$.

Check Minimum Variance Solution

If $\mathcal{M}$ is feasible, then compute Σ^{-1} and the minimum variance solution

$$w_{\text{min-var}} = \frac{\Sigma^{-1}\mathrm{e}}{\mathrm{e}^T\Sigma^{-1}\mathrm{e}} .$$

If $\mathrm{m}^T w_{\text{min-var}} \geq \mu_b$, then $w_{\text{min-var}}$ solves the problem $\mathcal{M}$.

Compute Two Portfolio Solution

If the problem is feasible and $w_{\text{min-var}}$ is not the solution, then compute the market weights

$$w_{\text{mk}} = \frac{\Sigma^{-1}\mathrm{m}}{\mathrm{e}^T\Sigma^{-1}\mathrm{m}} ,$$

and form the vector

$$v = w_{\text{mk}} - w_{\text{min-var}} .$$

The solution to $\mathcal{M}$ is then of the form

$$w = w_{\text{min-var}} + \alpha(w_{\text{mk}} - w_{\text{min-var}}) = w_{\text{min-var}} + \alpha v.$$

To determine α use the identity $\mathrm{m}^T w = \mu_b$ to get

$$\alpha = \frac{\mu_b - \mathrm{m}^T w_{\text{min-var}}}{\mathrm{m}^T v}.$$

It is remarkable that every solution to the Markowitz problem $\mathcal{M}$ can be represented as a linear combination of only two portfolios. These being the the minimum variance portfolio with weights $w_{\text{min-var}}$ and our *market* portfolio with weights w_{mk}. In the next section we will show that this is a general principal regardless of whether Σ is invertible or not.

3. The Efficient Frontier and the Two-Fund Theorem

In practice, one would like to have a better understanding of the return-risk trade-off since we want to both maximize return while minimizing risk. An alternative strategy is to try to balance these two objectives in a single objective function. One way to do this is to solve the QP

$$\begin{array}{lll} \mathcal{M}_\lambda & \text{minimize} & \frac{1}{2}w^T\Sigma w - \lambda \mathrm{m}^T w \\ & \text{subject to} & \mathrm{e}^T w = 1 \ . \end{array}$$

Observe that for $\lambda > 0$ the term $-\lambda \mathrm{m}^T w$ tries to push $\mathrm{m}^T w$ upwards to counter balance the downward pull of the term $\frac{1}{2}w^T\Sigma w$. The upward push on $\mathrm{m}^T w$ increases as λ is increased. The KKT conditions for the QP $\mathcal{M}_\lambda$ are

$$0 = \Sigma w - \lambda m - \gamma \mathrm{e} \tag{6}$$

$$\mathrm{e}^T w = 1 \ . \tag{7}$$

Note that these conditions are quite similar to the conditions (1)-(3) except that we no longer require that $\mathrm{m}^T w = \mu_b$. However, if $\bar{w}$ solves $\mathcal{M}_\lambda$ and we set $\mu_b = \mathrm{m}^T\bar{w}$, then the solution sets of $\mathcal{M}_\lambda$ and $\mathcal{M}$ coincide!

Proceeding as in the case of $\mathcal{M}$ under the assumption that Σ is invertible, we find that the solution to (6)-(7) is given by

$$\gamma = \frac{1 - \lambda \mathrm{m}^T\Sigma^{-1}\mathrm{e}}{\mathrm{e}^T\Sigma^{-1}\mathrm{e}}$$

and

$$\begin{aligned} w_\lambda &= \frac{\Sigma^{-1}\mathrm{e}}{\mathrm{e}^T\Sigma^{-1}\mathrm{e}} + \alpha\left[\frac{\Sigma^{-1}\mathrm{m}}{\mathrm{m}^T\Sigma^{-1}\mathrm{e}} - \frac{\Sigma^{-1}\mathrm{e}}{\mathrm{e}^T\Sigma^{-1}\mathrm{e}}\right] \\ &= (1-\alpha)\frac{\Sigma^{-1}\mathrm{e}}{\mathrm{e}^T\Sigma^{-1}\mathrm{e}} + \alpha\frac{\Sigma^{-1}\mathrm{m}}{\mathrm{m}^T\Sigma^{-1}\mathrm{e}} \\ &= (1-\alpha)w_{\text{min-var}} + \alpha w_{\text{mk}} \ , \end{aligned}$$

where $w_{\text{min-var}}$ and w_{mk} are as defined in the previous section and

$$\alpha = \lambda(\mathrm{m}^T\Sigma^{-1}\mathrm{e}) \ .$$

This gives a value for μ_b of

$$\mu_b = \mathrm{m}^T w_\lambda = \mu_{\text{min-var}} + \lambda \frac{\delta}{\mathrm{e}^T \Sigma^{-1} \mathrm{e}} \ ,$$

where

$$\delta = (\mathrm{e}^T \Sigma^{-1} \mathrm{e})(\mathrm{m}^T \Sigma^{-1} \mathrm{m}) - (\mathrm{m}^T \Sigma^{-1} \mathrm{e})^2 \ .$$

For $\lambda = 0$, we get, as expected, $\mu_{\text{min-var}}$, while as $\lambda \uparrow \infty$ we see that $\mu_b \uparrow \infty$ if $0 < \delta = (\mathrm{e}^T \Sigma^{-1} \mathrm{e})(\mathrm{m}^T \Sigma^{-1} \mathrm{m}) - (\mathrm{m}^T \Sigma^{-1} \mathrm{e})^2$. That is, if $0 < \delta$, the solution to $\mathcal{M}_\lambda$ traces out all possible solutions to $\mathcal{M}$ for all possible values of μ_b as λ moves from 0 to $+\infty$. Thus, in order to completely understand the risk-return trade-off, we need only graph the curve

$$(\sqrt{(w_\lambda^T \Sigma w_\lambda)}\,,\, r_\lambda) \;=\; (\sqrt{(\mathrm{var}(r_\lambda))}\,,\, E(r_\lambda)),$$

where

$$r_\lambda = w_\lambda^T r \ ,$$

as λ varies from 0 to $+\infty$. This is very reminiscent of a mean-standard deviation curve! Indeed, we will see that it is exactly this concept! In the context of Markowitz mean-variance portfolio theory, this is called the *efficiency curve* or *efficient frontier*. Any portfolio associated with a point on the efficient frontier is called and efficient portfolio.

In order to make this connection clear, let us consider the mean-standard deviation diagram for the entire portfolio. That is, we graph all pairs (σ_i, μ_i) for each asset $i = 1, 2, \ldots, n$ where $\sigma_i^2 = \mathrm{var}(r_i)$ and $\mu_i = E(r_i)$. We then define the *feasible* region to be the set of possible pairs of the form

$$(\sqrt{\mathrm{var}(w^T z)}, E(w^T z))$$

for all possible values of $w \in \mathbb{R}^n$ satisfying $\mathrm{e}^T w = 1$. The upper boundary of this region is precisely the efficient frontier! What is perhaps more remarkable is that the outer boundary is itself a hyperbola given as the mean-standard deviation diagram of two portfolios.

Theorem 3.1. *(The Two-Fund Theorem) The efficient frontier for $\mathcal{M}_\lambda$ as λ travels from 0 to $+\infty$ is the upper half of the mean-standard deviation diagram for two efficient portfolios.*

Proof. Let $\mu_b^{(1)}$ and $\mu_b^{(2)}$ satisfy $\mu_{\text{min-var}} < \mu_b^{(1)},\ \mu_b^{(2)}$ with $\mu_b^{(1)} < \mu_b^{(2)}$. Consider the KKT conditions (1)-(3). Since neither $\mu_b^{(1)}$ and $\mu_b^{(2)}$ equals $\mu_{\text{min-var}}$, the constraint $\mathrm{m}^T w \geq \mu_b^{(i)}$ is active for both $i = 1$ and $i = 2$.

Therefore, the KKT conditions become

$$0 = \Sigma w - \lambda \mathrm{m} - \gamma \mathrm{e}\ , \tag{8}$$

$$\mu_b = \mathrm{m}^T w\ , \quad \text{and} \tag{9}$$

$$1 = \mathrm{e}^T w. \tag{10}$$

This is just a linear system of equations for which a solution is always guaranteed to exist for all values of $\mu_b > \mu_{\text{min-var}}$. If $(w^i, \lambda_i, \gamma_i)$ are such solutions for $\mu_b = \mu_b^{(i)},\ i = 1, 2$, then

$$(w^\alpha, \lambda_\alpha, \gamma_\alpha) = (1-\alpha)(w^1, \lambda_1, \gamma_1) + \alpha((w^2, \lambda_2, \gamma_2)$$

solves (8)-(10) for $\mu_b = (1-\alpha)\mu_b^{(1)} + \alpha\mu_b^{(2)}$ for all values of $\alpha \in I\!R$. Define the portfolios $r_{(1)} = z^T w^1$, $r_{(2)} = z^T w^2$, and $r_\alpha = (1-\alpha)r_{(1)} + \alpha r_{(2)}$ for all $\alpha \in I\!R$. Then, by construction, as α travels from $\frac{\mu_{\text{min-var}} - \mu_b^{(1)}}{\mu_b^{(2)} - \mu_b^{(1)}}$ to $+\infty$ the pair $(\sqrt{\mathrm{var}(r_\alpha)}, E(r_\alpha))$ traces out both the efficient frontier and the mean-standard deviation curve for the efficient portfolios $r_{(1)}$ and $r_{(2)}$. $\square$

Thus, according to Markowitz mean-variance theory, only two mutual funds are required to for any investor to achieve their desired balance between return and risk.

4. The Effect of a Risk-Free Asset

In the previous section, we assumed that all assets in the portfolio were *risky* and that the covariance matrix of their returns was nonsingular. However, in practice, there are assets whose risk level is so low that we model them as *risk free*. Treasury bills are an example of a risk free asset.

We now repeat the Markowitz mean-variance analysis with the inclusion of an asset f having a risk free return r_f. Since the asset is risk free we also know that its variance is zero and its covariance with any other asset is zero as well. If we write our random vector of returns as $\hat{x} = (r_f, x^T)^T$ where $x = (r_1, r_2, \ldots, r_n)^T$ and r_i is the rate of return on risky asset i, then the covariance matrix for $\hat{x}$ is

$$\hat{\Sigma} = \begin{bmatrix} 0 & 0 \\ 0 & \Sigma \end{bmatrix},$$

where Σ is the covariance matrix for x. Therefore, the Markowitz QP can be written as

$$\begin{array}{lll} \mathcal{M}_f: & \text{minimize} & \frac{1}{2} w^T \Sigma w \\ & \text{subject to} & r_f w_0 + \mathrm{m}^T w \geq \mu_b,\ \text{and}\ w_0 + \mathrm{e}^T w = 1\ . \end{array}$$

Here w_0 is the weight to be assigned to the risk free asset, μ_b is the baseline rate of return as before, and the vectors m and e are to be interpreted as previously. Since we can always achieve a rate of return on our investments of r_f, we assume that $\mu_b \geq r_f$. Also we continue to assume that Σ is nonsingular. Note that if $w_0 > 0$, then we are investing at the risk free rate, while if $w_0 <$ we are borrowing at the risk free rate.

The KKT conditions for $\mathcal{M}_f$ are

$$
\begin{aligned}
&(11) \qquad & 0 &= \lambda r_f + \gamma \\
&(12) \qquad & 0 &= \Sigma w - \lambda \mathrm{m} - \gamma \mathrm{e} \\
&(13) \qquad & 0 &= \lambda(r_f w_0 + \mathrm{m}^T w - \mu_b) \\
&(14) \qquad & 1 &= w_0 + \mathrm{e}^T w \\
&(15) \qquad & \mu_b &\leq r_f w_0 + \mathrm{m}^T w \\
&(16) \qquad & 0 &\leq \lambda \;.
\end{aligned}
$$

Again we consider the two cases where the constraint $\mu_b \leq r_f w_0 + \mathrm{m}^T w$ is either inactive or active. If it is inactive (i.e., $\mu_b < r_f w_0 + \mathrm{m}^T w$), then by (13) $\lambda = 0$ since it is non-negative. But then (11) implies that $\gamma = 0$, which in turn implies that $w = 0$ by (12). In this case, the solution is given by $w_0 = 1$, that is, the *optimal* portfolio consists of the risk free asset alone.

Next, if the constraint $\mu_b \leq r_f w_0 + \mathrm{m}^T w$ is active at the solution, then we can proceed as before to derive the optimal solution. First multiply (12) by Σ^{-1} and apply (11) to get

$$w = \lambda \Sigma^{-1}(\mathrm{m} - r_f \mathrm{e}) \;. \qquad (17)$$

Using (14) and equality in (15), we get the two equations

$$
\begin{aligned}
1 - w_0 &= \mathrm{e}^T w = \lambda \mathrm{e}^T \Sigma^{-1}(\mathrm{m} - r_f \mathrm{e}) \quad \text{and} \\
\mu_b - r_f w_0 &= \mathrm{m}^T w = \lambda \mathrm{m}^T \Sigma^{-1}(\mathrm{m} - r_f \mathrm{e}) \;.
\end{aligned}
$$

This can be rewritten as the 2×2 system of equations

$$\begin{bmatrix} 1 & \mathrm{e}^T \Sigma^{-1}(\mathrm{m} - r_f \mathrm{e}) \\ r_f & \mathrm{m}^T \Sigma^{-1}(\mathrm{m} - r_f \mathrm{e}) \end{bmatrix} \begin{pmatrix} w_0 \\ \lambda \end{pmatrix} = \begin{pmatrix} 1 \\ r_f \end{pmatrix} \;.$$

Solving this system by Gaussian elimination gives

$$\begin{pmatrix} w_0 \\ \lambda \end{pmatrix} = \begin{pmatrix} 1 - (\mu_b - r_f) \frac{\mathrm{e}^T \Sigma^{-1}(\mathrm{m} - r_f \mathrm{e})}{(\mathrm{m} - r_f \mathrm{e})^T \Sigma^{-1}(\mathrm{m} - r_f \mathrm{e})} \\ \frac{(\mu_b - r_f)}{(\mathrm{m} - r_f \mathrm{e})^T \Sigma^{-1}(\mathrm{m} - r_f \mathrm{e})} \end{pmatrix} \;,$$

if $\mathrm{m} \neq r_f\mathrm{e}$ (if $\mathrm{m} = r_f\mathrm{e}$, the $w_0 = 1$ and $w = 0$). Plugging this into (17), we get

$$\begin{aligned}
\begin{pmatrix} w_0 \\ w \end{pmatrix} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \alpha\left[\begin{pmatrix} 1 - \mathrm{e}^T\Sigma^{-1}(\mathrm{m} - r_f\mathrm{e}) \\ \Sigma^{-1}(\mathrm{m} - r_f\mathrm{e}) \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right] \\
&= (1-\alpha)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \alpha\begin{pmatrix} 1 - \mathrm{e}^T\Sigma^{-1}(\mathrm{m} - r_f\mathrm{e}) \\ \Sigma^{-1}(\mathrm{m} - r_f\mathrm{e}) \end{pmatrix}, \qquad (18)
\end{aligned}$$

where

$$\alpha = \frac{(\mu_b - r_f)}{(\mathrm{m} - r_f\mathrm{e})^T\Sigma^{-1}(\mathrm{m} - r_f\mathrm{e})} .$$

That is, the weights for the optimal portfolio is a linear combination of the two sets of weights

$$w_f = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad w_{\mathrm{M}} = \begin{pmatrix} 1 - \mathrm{e}^T\Sigma^{-1}(\mathrm{m} - r_f\mathrm{e}) \\ \Sigma^{-1}(\mathrm{m} - r_f\mathrm{e}) \end{pmatrix} .$$

This observation yields the following result.

Theorem 4.1. *(The One Fund Theorem)*
If the selection of assets for investment includes a risk free asset, then there exists a single fund F of risky assets such that every efficient portfolio can be constructed as a linear combination of the risk free asset and the fund F.

5. The Capital Asset Pricing Model (CAPM)

A consequence of the one fund theorem is that if a risk free asset is one of the assets that we can select from, then the efficient frontier in the Markovitz mean-variance model is a straight line. Using the representation for the optimal weights given in (18), we can give a formula for this line. Begin by defining

$$r_{\mathrm{M}} = \begin{pmatrix} r_f \\ r \end{pmatrix}^T \begin{pmatrix} 1 - \mathrm{e}^T\Sigma^{-1}(\mathrm{m} - r_f\mathrm{e}) \\ \Sigma^{-1}(\mathrm{m} - r_f\mathrm{e}) \end{pmatrix}$$

to be the *market* portfolio, and let $r = w_0 r_f + r^T w$ be any efficient portfolio so that the weights $(w_0, w^T)^T$ satisfy (18) for some value of α. Then μ_r, the expected return for the portfolio r, satisfies the equation

$$\mu_r = r_f + \frac{\mu_{\mathrm{M}} - r_f}{\sigma_{\mathrm{M}}}\sigma_r,$$

where $\mu_r = E(r)$, $\sigma_r^2 = \mathrm{var}(r)$,

$$\begin{aligned}
\mu_{\mathrm{M}} &= E(r_{\mathrm{M}}) = r_f + (\mathrm{m} - r_f\mathrm{e})^T\Sigma^{-1}(\mathrm{m} - r_f\mathrm{e}), \text{ and} \\
\sigma_{\mathrm{M}}^2 &= \mathrm{var}(r_{\mathrm{M}}) = (\mathrm{m} - r_f\mathrm{e})^T\Sigma^{-1}(\mathrm{m} - r_f\mathrm{e}) .
\end{aligned}$$

This line gives the efficient frontier.

Let us now consider the case where the choice of assets in our Markowitz model consists of the entire market of all trade-able securities. In this case, the line given above to describe the efficient frontier is called the *capital market line*. The slope of the line $\frac{\mu_{\mathrm{M}}-r_f}{\sigma_{\mathrm{M}}}$ is called the *price of risk*. We now consider the implications of this line for the pricing of assets.

Let i be any asset and consider a portfolio consisting of i and the market portfolio r_{M} alone. The mean-standard deviation curve for this combination must lie entirely below the capital market line and yet touches this line at the portfolio r_{M}. Consequently, the capital market line must be tangent to the mean-standard deviation curve for i and r_{M} at r_{M}. This observation gives the following important result.

Theorem 5.1. *(The Capital Asset Pricing Model (CAPM)) The expected return on any asset i, μ_i, satisfies*

$$\mu_i = r_f + \beta_i(\mu_M - r_f), \tag{19}$$

where

$$\beta_i \ = \ \frac{\sigma_{iM}}{\sigma_M^2}$$

and σ_{iM} is the covariance of the return on asset i and the market portfolio r_M.

Proof. The formula (19) follows from the tangency arguement given before the statement of the theorem. Let r_i be the return on the asset under consideration, and consider the protfolio

$$r_\alpha = \alpha r_i + (1-\alpha) r_{\mathrm{M}}.$$

The expected return on this portfolio is

$$\mu_\alpha = \alpha\mu_i + (1-\alpha)\mu_{\mathrm{M}},$$

and the variance is

$$\sigma_\alpha^2 = \alpha^2\sigma_i^2 + 2\alpha(1-\alpha)\sigma_{i\mathrm{M}} + (1-\alpha)^2\sigma_{\mathrm{M}}^2\ .$$

Differentiating these equations with respect to α gives

$$\frac{dr_\alpha}{d\alpha} = (r_i - r_{\mathrm{M}})$$

and

$$\frac{d\sigma_\alpha}{d\alpha} = \frac{1}{\sigma_\alpha}\left[\alpha\sigma_i^2 + (1-2\alpha)\sigma_{i\mathrm{M}}) + (\alpha-1)\sigma_{\mathrm{M}}^2\right]\ .$$

Therefore,

$$\frac{dr_\alpha}{\sigma_\alpha} = \frac{dr_\alpha/d\alpha}{d\sigma_\alpha/d\alpha} = \frac{\sigma_\alpha(r_i - r_{\mathrm{M}})}{[\alpha\sigma_i^2 + (1-2\alpha)\sigma_{i\mathrm{M}}) + (\alpha-1)\sigma_{\mathrm{M}}^2]}\ .$$

Since the mean-standard deviation curve for r_i and r_{M} is tangent to the capital market line at $\alpha = 0$, we get the formula

$$\frac{\mu_{\mathrm{M}} - r_f}{\sigma_{\mathrm{M}}} = \left.\frac{dr_\alpha}{\sigma_\alpha}\right|_{\alpha=0} = \frac{\sigma_{\mathrm{M}}(r_i - r_{\mathrm{M}})}{\sigma_{i\mathrm{M}} - \sigma_{\mathrm{M}}^2} .$$

Solving for μ_{M} gives

$$\begin{aligned} \mu_{\mathrm{M}} &= r_{\mathrm{M}} + \frac{(\mu_{\mathrm{M}} - r_f)}{\sigma_{\mathrm{M}}} \frac{(\sigma_{i\mathrm{M}} - \sigma_{\mathrm{M}}^2)}{\sigma_{\mathrm{M}}} \\ &= r_f + \frac{(\mu_{\mathrm{M}} - r_f)}{\sigma_{\mathrm{M}}^2} \sigma_{i\mathrm{M}} \\ &= r_f + \beta_i(\mu_{\mathrm{M}} - r_f) . \end{aligned}$$

□

The value β_i in Theorem 5.1 is referred to as the *beta* of the asset i. The value $(\mu_i - r_f)$ is called the *excess rate of return* of the asset. The beta of an asset tells the the excess rate of return of an asset as a presentage of the market excess rate of return. An asset with a beta value less than 1 should be a conservative investment in that its variance should be less than that of the market. On the other hand, if an asset's beta exceeds 1, then the asset should be riskier than the market in the sense that its variance should excedd that of the market.

The CAPM can be used to price an asset. In order to see how this is done consider an asset that is purchased for the price P and later sold at the price Q. We model the sale price Q as a random variable with mean μ_Q. The rate of return and expected rate of return for this asset are

$$r = \frac{Q - P}{P} \quad \text{and} \quad \mu_r = \frac{\mu_Q - P}{P} .$$

Plugging this into the formula (19) gives the relation

$$\frac{\mu_Q - P}{P} = \mu_r = r_f + \beta_r(\mu_{\mathrm{M}} - r_f),$$

or equivalently,

$$P = \frac{\mu_Q}{1 + r_f + \beta_r(\mu_{\mathrm{M}} - r_f)}, \tag{20}$$

where β_r is the beta of the asset. Note that β_r satisfies

$$\begin{aligned}\beta_r &= \frac{\sigma_{r\mathrm{M}}}{\sigma_{\mathrm{M}}^2} \\ &= \frac{\mathrm{cov}(r, r_{\mathrm{M}})}{\mathrm{var}(r_{\mathrm{M}})} \\ &= \frac{\mathrm{cov}(\frac{Q-P}{P}, r_{\mathrm{M}})}{\sigma_{\mathrm{M}}^2} \\ &= \frac{\mathrm{cov}(\frac{Q}{P} - 1, r_{\mathrm{M}})}{\sigma_{\mathrm{M}}^2} \\ &= \frac{\mathrm{cov}(Q, r_{\mathrm{M}})}{P\sigma_{\mathrm{M}}^2} .\end{aligned}$$

Therefore,

$$P = \frac{\mu_Q}{1 + r_f + \frac{\mathrm{cov}(Q, r_{\mathrm{M}})}{P\sigma_{\mathrm{M}}^2}(\mu_{\mathrm{M}} - r_f)} ,$$

or equivalently,

$$1 = \frac{\mu_Q}{P(1 + r_f) + \mathrm{cov}(Q, r_{\mathrm{M}})(\mu_{\mathrm{M}} - r_f)/\sigma_{\mathrm{M}}^2} .$$

Resolving for P gives

$$P = \frac{1}{1 + r_f}\left[\mu_Q - \frac{\mathrm{cov}(Q, r_{\mathrm{M}})(\mu_{\mathrm{M}} - r_f)}{\sigma_{\mathrm{M}}^2}\right] . \tag{21}$$

We call the formula (20) the *CAPM pricing formula*. It gives the price of an asset as a function of its beta and expected return. The formula (21) is called the *certainty equivalent pricing formula*. This formula does not require knowledge of the asset's beta but does require knowledge of the ratio

$$\frac{\mathrm{cov}(Q, r_{\mathrm{M}})}{\sigma_{\mathrm{M}}^2}.$$

Note that the certainty equivalent pricing formula demonstrates that the purchase price P is a linear function of the sale price Q.

The term

$$\left[\mu_Q - \frac{\mathrm{cov}(Q, r_{\mathrm{M}})(\mu_{\mathrm{M}} - r_f)}{\sigma_{\mathrm{M}}^2}\right]$$

appearing in the certainty equivalent pricing formula is called the *certainty equivalent* of the random variable Q. It is a fixed an certain amount that can be combined with the risk free discount factor $(1 + r_f)^{-1}$ to obtain the price of the asset P.

Exercises

(1) Suppose there are n assets which are uncorrelated (they might be n different "wild cat" oil well prospects). You may invest in any one, or in any combination of them. The mean rate of return μ is the same for each asset, but the variances are different. The return on asset i has variance σ_i^2 for $i = 1, 2, \ldots, n$.
   (a) Describe the efficient set for this situation.
   (b) Write a formula for the minimum-variance point. Express your result in terms of

$$\hat{\sigma}^2 = \left(\sum_{i=1}^{n} \frac{1}{\sigma_i^2} \right) .$$

(2) An events planner has 1 million dollars to invest in an outdoor concert. It is expected that the concert will yield 3 million dollars on the \$1 million investment, unless it rains. If it rains, the entire \$1 million investment is lost. There is a 50 percent chance that it will rain on the day of the concert. However, the planner can buy rain insurance for 50 cents on the dollar. That is, for each 50 cents of rain insurance, the planner will receive \$1 if it rains. The planner can purchase as much rain insurance as desired up to a face value of 3 million dollars.
   (a) What is the expected rate of return on this investment if the planner buys u dollars of insurance?
   (b) How much insurance should be purchased to minimize the variance in the return on this investment? What is the minimum variance, and what is the associated expected return?

(3) Consider 3 assets with rates of return r_1, r_2, and r_3, respectively. The covariance matrix and expected rates of return are

$$\Sigma = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \quad \text{and} \quad \mathrm{m} = \begin{bmatrix} .4 \\ .4 \\ .8 \end{bmatrix} .$$

   (a) Find the minimum variance portfolio.
   (b) Find a second efficient portfolio.
   (c) If the risk free rate is $r_f = .2$, find an efficient portfolio of risky assets.

(4) It is often impractical to use all of the assets that are incorporated into a specified portfolio (such as the efficient market portfolio). One alternative is to find a portfolio composed of

a given set of n stocks that tracks the specified portfolio most closely in the sense of minimizing the variance of the difference in returns.

Specifically, suppose that the target portfolio has a random rate of return of $r_{\rm M}$. Further, suppose that there are n assets with random rates of return $r_1, r_2, \ldots, r_n$. We wish to find the portfolio having rate of return

$$r = w_1 r_1 + w_2 r_2 + \cdots + w_n r_n$$

that minimizes the variance of $r - r_{\rm M}$:

$$\begin{array}{lll} \mathcal{T}: & \text{minimize} & \frac{1}{2}\text{var}(w_1 r_1 + w_2 r_2 + \cdots + w_n r_n - r_{\rm M}) \\ & \text{subject to} & \mathrm{e}^T w = 1\ . \end{array}$$

Let $x = (r_1, r_2, \ldots, r_n)^T$ be the random vector of returns. Set $\Sigma = \text{cov}(x)$, $s = (\text{cov}(r_1, r_{\rm M}), \text{cov}(r_2, r_{\rm M}), \ldots \text{cov}(r_n, r_{\rm M}))^T$ and $\sigma_{\rm M}^2 = \text{var}(r_{\rm M})$. Assume that Σ is an invertable matrix.

(a) Use Σ, s, and $\sigma_{\rm M}^2$ to write a matrix expression for

$$\text{var}(w_1 r_1 + w_2 r_2 + \cdots + w_n r_n - r_{\rm M}).$$

(b) Write down the KKT conditions for this quadratic program.

(c) Use the KKT conditions to compute an expression for the solution to the QP $\mathcal{T}$.

(d) Although this portfolio tracks the desired portfolio most closely in terms of variance, it may sacrifice the mean. Hence a logical approach is to minimize the variance of the tracking error subject to achieving a given mean return μ_b. As the mean μ_b is varied, this results in a family of portfolios that are efficient in a new sense–say, tracking efficient. The QP for this new problem has the form

$$\begin{array}{lll} \mathcal{T}_{\mu_b}: & \text{minimize} & \frac{1}{2}\text{var}(w_1 r_1 + w_2 r_2 + \cdots + w_n r_n - r_{\rm M}) \\ & \text{subject to} & \mathrm{e}^T w = 1 \text{ and } \mathrm{m}^T w \geq \mu_b\ , \end{array}$$

where $\mathrm{m} = E(x)$.

(i) What are the KKT conditions for this new QP?

(ii) Use these KKT conditions to compute an expression for the solution to the QP $\mathcal{T}_{\mu_b}$.

(iii) Is there and analogue of the Two Fund Theorem for the tracking efficient frontier? If so, give an expression for two funds that can be used to obtain every tracking efficient portfolio.

(5) Assume that the expected rate of return on the market portfolio is 23% ($r_{\mathrm{M}} = .23$) and the rate of return on T-Bills (risk free rate) is 7% ($r_f = .07$). The standard deviation of the market is 32% ($\sigma_{\mathrm{M}} = .32$). Assume that the market portfolio is efficient.
  (a) What is the equation for the capital market line?
  (b) If an expected return of 39% is desired, what is the standard deviation of this position?
  (c) If you have \$1000 to invest, how should you invest it to achieve the above position?
  (d) If you invest \$300 at the risk free rate and \$700 in the market portfolio, how much money do you expect to have at the end of the year?

(6) Let $w_m = w_{\text{min-var}}$ denote the weights for a set of risky assets corresponding to the minimum variance point. Let w_τ be the weights for any other portfolio on the efficient frontier for this set of assets. Define r_m and r_τ to be the corresponding returns.
  (a) There is a formula of the form

$$\sigma_{m\tau} = A\sigma_m^2,$$

  where $\sigma_{m\tau} = \mathrm{cov}(r_m, r_\tau)$ and $\sigma_m^2 = \mathrm{var}(r_m)$. Find A. [*Hint*: Consider the portfolios with weights $(1-\alpha)w_m + \alpha w_\tau$, and consider small variations of the variance of such portfolios near $\alpha = 0$.]
  (b) Corresponding to the portfolio w_τ there is a portfolio w_z on the minimum variance set (not necessarily the efficient frontier) that has zero beta with respect to w_τ: that is, $\sigma_{z\tau} = 0$. This portfolio can be expressed as $w_z = (1 - \alpha)w_m + \alpha w_\tau$. Find the proper value of α.
  (c) Show the relationship between these three portfolios on a diagram that includes the feasible region.

(7) Consider an oil drilling venture. The price of a share in this venture is \$800 with an expected yield after 1 year of \$1000 per share. However, due to the high uncertainty about how much oil is at the drilling site, the standard deviation of the rate of return on this investment is $\sigma = .4$. Currently the risk free rate is .1. The expected rate of return on the market portfolio is .17 with a standard deviation of .12. The beta of the drilling shares is .6. What is the value of the drilling shares based in the CAPM? Would you advise purchasing these shares based on this model?