Math 408
Cash Flow Streams, Present Value, and Internal Rate of Return

1. Cash Flow Streams

In investment science, notions of investment are based on the concept of *flows* of expenditures and receipts over time. The goal is to tailor the pattern of these flows over time so that they satisfy some underlying well defined objective such as the maximization of the returns on investment. In this course we will think of the undelying expenditures and receipts as denominated in cash (currency). In this case, the net of expenditures and receipts in a given time period is called the *cash flow* in that period. The series of cash flows over a chosen number of time periods (or time horizon) is called the cash flow stream for that series of time periods. The units of time used will vary with the problem under consideration. In some instances it will be months, others quarters, and others years. In each setting the time period and the time horizon to be considered must be precisely specified. We will describe cash flow streams as vectors of real numbers. Abstractly, such a vector can be written as

$$x = (x_0, x_1, x_2, \ldots, x_n),$$

where n is the number of time periods. If the time period is months and the stream horizon is 10 years, then $n = 12 \times 10 = 120$.

Knowing the basic properties of geometric series is extremely helpful in the analysis of cash flow streams. We now take a moment to review these properties.

Geometric Series:

Consider a series of the form

$$S_n = \sum_{k=0}^{n} \rho^k,$$

where ρ is any real number. Such a series is called a *geometric series*. It is said to be an infinite geometric series if we take $n = \infty$, in which case the collection of symbols $\sum_{k=0}^{\infty} \rho^k$ is to interpreted as the limit

$$\sum_{k=0}^{\infty} \rho^k := \lim_{n\to\infty} \sum_{k=0}^{n} \rho^k.$$

To sum this series simply observe that

$$\begin{aligned} S_n &= 1 + \rho + \rho^2 + \rho^3 + \cdots + \rho^n \\ &= 1 + \rho\,[\,1 + \rho + \rho^2 + \rho^3 + \cdots + \rho^{n-1}\,] \\ &= 1 + \rho\,[\,1 + \rho + \rho^2 + \rho^3 + \cdots + \rho^n\,] \;-\; \rho^{n+1} \\ &= 1 + \rho S_n - \rho^{n+1}. \end{aligned}$$

Therefore,

$$(1-\rho)S_n = 1 - \rho^{n+1},$$

or equivalently

$$S_n = \frac{1-\rho^{n+1}}{1-\rho}.$$

If $|\rho| < 1$, then $\lim_{n\to\infty} \rho^n = 0$, and so

$$S_\infty = \lim_{n\to\infty} \sum_{k=1}^{n} \rho^k = \lim_{n\to\infty} \frac{1-\rho^{n+1}}{1-\rho} = \frac{1}{1-\rho}.$$

It is also usefull to consider the related series

$$\hat{S}_n = \sum_{k=1}^{n} \rho^k = S_n - 1.$$

The formulas for S_n given above imply that

$$\hat{S}_n = \frac{1-\rho^{n+1}}{1-\rho} - 1 = \frac{\rho-\rho^{n+1}}{1-\rho} = \rho\frac{1-\rho^n}{1-\rho} = \rho S_{n-1}.$$

1.1. **Compound interest.** Consider a simple savings account with initial investment P dollars invested at an annual interest rate of r compounded monthly and closed after n years. After the first month the account will contain the original principal P plus the interest earned on that principal over the month $\frac{r}{12}P$ dollars. That is, the account will contain $(1+\frac{r}{12})P$ dollars. At the end of the second month the account will earn interest on $(1+\frac{r}{12})P$ dollars since this is the principal for that month. Hence, as before, the content of the account at the end of the second month is the sum of the principal at the begining of the month and the interest earned on this principal during the month which is

$$\frac{r}{12}(1+\frac{r}{12})P + (1+\frac{r}{12})P = (1+\frac{r}{12})^2 P.$$

Continuing in this way we see that the content of the account after n years is $(1+\frac{r}{12})^{(12n)}P$.

We now describe a cash flow stream associated with this investment. The time period for the stream is one month, and the horizon for the stream is N years or $12N$ periods. In the first period we make an investment of P dollars which is the initial principal. Hence we set

$x_0 = -P$. Of the remaining periods, the only period during which money is received from the account is in the last period when the account is closed. Hence $x_{12N} = (1 + \frac{r}{12})^{(12N)}P$. Hence the cash flow stream for this investment is a vector of length $12N+1$ with $x_0 = -P$, $x_k = 0,\ k = 1, 2, \ldots, 12N - 1$ and $x_{12N} = (1 + \frac{r}{12})^{(12N)}P$.

There is another equally valuable way to view this cash flow stream. In this view we consider the initial principal P as the purchase price for the stream and so do not included it in the stream itself. In this view the cash flow stream is again of length $12N + 1$ but now all entries in the stream are zero except for the last which is $x_{12N} = (1 + \frac{r}{12})^{(12N)}P$. In general, there are many ways to model the cash flow stream for the same investment opportunity. The choice of model depends on the questions being asked.

Example 1.1. *(***Home Mortgages***)*
A home mortgage is a loan from a bank or broker to facilitate the purchase of a home. Some of the key features of a standard home mortgage are (a) the principal borrowed, P_0, (b) the annual interest rate at which it is borrowed, r, (c) and the monthly payment, y, and (d) the number of years to pay off the loan, N. We assume that the payment and compounding periods coincide (this is almost always the case). Clearly, the values of the variables $P_0, r, y,$ and N are related in a special way so that the loan can be payed off in n years. If we denots P_k as the principal remaining after k periods, then we must have $P_{12N} = 0$. This equation can be used to determine the relationship between the variables $P_0, r, y,$ and N if we can write P_{12N} as a function of $P_0, r, y,$ and N. To begin with note that for each period k we have

$$P_k = (1 + \frac{r}{12})P_{k-1} - y\ .$$

This recursion gives

$$\begin{aligned}
P_k &= (1 + \frac{r}{12})P_{k-1} - y \\
&= (1 + \frac{r}{12})((1 + \frac{r}{12})P_{k-2} - y) - y \\
&= (1 + \frac{r}{12})^2 P_{k-2} - y - (1 + \frac{r}{12})y \\
&= (1 + \frac{r}{12})^3 P_{k-3} - y - (1 + \frac{r}{12})y - (1 + \frac{r}{12})^2 y \\
&\ \vdots
\end{aligned}$$

$$
\begin{aligned}
&= (1+\frac{r}{12})^k P_0 - y - \cdots - (1+\frac{r}{12})^{k-1} y \\
&= (1+\frac{r}{12})^k P_0 - y \sum_{j=0}^{k-1} (1+\frac{r}{12})^j \\
&= (1+\frac{r}{12})^k P_0 - y \frac{1-(1+\frac{r}{12})^k}{1-(1+\frac{r}{12}))} \\
&= (1+\frac{r}{12})^k P_0 - y \frac{12}{r}((1+\frac{r}{12})^k - 1).
\end{aligned}
$$

The penultimate equality in this derivation makes use of the fundamental identity for geometric series:

$$\sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x}.$$

Therefore, if $P_{12N} = 0$*, we obtain the identity*

$$(1+\frac{r}{12})^{12N} P_0 = y \frac{12}{r}((1+\frac{r}{12})^{12N} - 1). \tag{1}$$

In particular, given P_0 *and* r*, the monthly payments* y *are*

$$y = P_0 \frac{r}{12} \frac{(1+\frac{r}{12})^{12N}}{(1+\frac{r}{12})^{12N} - 1}.$$

Similarly, if you know the prevailing interest rate r *for home mortgages and you are willing to pay up to* y *dollars a month on you mortgage, then you can determine how much you are willing to borrow from the formula*

$$P_0 = y \frac{12}{r} \left(1 - (1+\frac{r}{12})^{-12N}\right).$$

The cash flow stream for this investment is

$$(P_0, -y, -y, \ldots, -y),$$

where the "$-y$*" term appears each period for* $12N$ *periods.*

1.2. **Continuous Compounding.** One can think of compounding the interest on a savings account more frequently than just monthly. For example, it might be compounded daily, or even more frequently. What happens as the number of times the rate is compounded goes to infinity? To answer this we must reach into our deep dark calculus past. Suppose we compound m times a year at an annual rate of r. The after t years such an account with an initial principal of P dollars will hold

$$(1+\frac{r}{m})^{mt} P \quad \text{dollars.}$$

Calculus tells us that

$$\lim_{m\to\infty}(1+\frac{r}{m})^m = e^r.$$

If we let $m \to \infty$ we say that the account is compounding *contiuously.* Such an account yields Pe^{rt} from an initial investment of P dollars invested at an annual rate r after t years (here t need not be given in whole years, e.g. $t = \sqrt{2}$ works just fine). Continuous compounding is often used to simplify the analysis in very complicated settings.

Interest rates are typically quoted at an annual rate regardless on the compounding schedule. This interest rate is called the *nominal* interest rate. The *effective* interest rate is the equivalent annually compounded rate. That is, it is the rate of return of the investment after one year. If an investment returns Q dollars after one year from an initial investment of P dollars, then the effective interest rate of the investment is

$$r_e = \frac{Q-P}{P}.$$

Therefore, an investment at an annual interest rate of 4 percent compounded continuously has an effective interest rate of approximately 4.081077419239 percent since

$$e^{.04} - 1 \approx .04081077419239 \quad .$$

1.3. **Present Value.** There are a wide variety of investment opportunities available in the market place. All of these can be modeled as having an associated cash flow stream (perhaps unknowable), but the period and time to maturity of each such stream can vary considerably. In order to compare any two investment opportunities, or equivalently, their cash flow streams, we need a measure that is independent of the period and time to maturity of the stream. One such measure is *present value.* In effect the present value of a cash flow stream is its total value in todays dollars. One way to compute present value is to assume a constant prevailing rate of return r. This prevailing rate of return is sometimes taken to be a *riskless* rate of return (for example a suitable treasury bill rate), or it can be a historical average rate of return. The choice can depend on the time to maturity of the underlying investments to be compared. Once a rate r has been established, one can then derive the corresponding per period rate of return of a given cash flow stream r_p. For example, if the rate is .04 (or 4 percent) and the period is months, then the per period rate is $r_p = (.04)/12$.

Let us now suppose we are given a cash flow stream $x = (x_0, x_1, x_2, \ldots, x_n)$ with a prevailing per period rate of return r_p. The present value of this stream can be computed by considering each flow element separately.

Consider flow element x_0. This is the initial cash flow element today, so its present value today is x_0. Next consider x_1. This is the cash flow one period from now. Take the present value of x_1 to be the value of the principal after one period if it were invested at a per period rate of r_p. That is, we need to solve the equation

$$x_1 = (1 + r_p)P_1$$

for P_1, where P_1 is the principal at the end of the first period. Obviously this yields a present value for x_1 equal to $\frac{x_1}{(1+r_p)}$. In general, the present value of the cash flow element x_k equals the principal required to obtain x_k after k periods if it were invested at a per period rate of r_p. That is, the present value of x_k is the value of P_k, the value of the principal at the end of the kth period, that satisfies the equation

$$x_k = (1 + r_p)^k P_k,$$

which is $\frac{x_k}{(1+r_p)^k}$.

Definition 1.2. *(***Present Value***) Given a cash flow stream $x = (x_0, x_1, \ldots, x_n)$ and a per period interest rate r_p, the present value of this stream relative to this rate is*

$$PV = x_0 + \frac{x_1}{(1 + r_p)} + \frac{x_2}{(1 + r_p)^2} + \cdots + \frac{x_n}{(1 + r_p)^n}.$$

The factor $d_p = \frac{1}{(1+r_p)}$ is called the discount factor for this stream.

Another way to describe present value is in terms of *discount factors*. In the definition given above, the factor $d_p = \frac{1}{(1+r_p)}$ is called the *discount factor* for this stream. Thus, we compute present value of a cash flow stream relative to a given per period discount factor.

Example 1.3. *(***Savings Accounts***)*
In the discussion of compound interest given above, we described two different cash cash flow streams for a savings account with initial principal invested at annual rate r compounded monthly and closed after N years. These are

$$\begin{aligned} x &= (-P, 0, \ldots, 0, (1 + r_p)^{12N} P), \qquad \textit{and} \\ y &= (0, \ldots, 0, (1 + r_p)^{12N} P), \end{aligned}$$

where $r_p = r/12$ and the second stream has the underlying assumption that the initial principal is to be interpreted as the purchase price of the stream. The present value of these streams relative to r_p is 0 and P, respectively. Thus, present value can be used to value the purchase price of a cash flow stream today. Why is the purchase price of x zero while that of y is P?

Example 1.4. *(***Home Mortgages***)*
Recall that the cash flow stream for a home mortgage with initial principal P_0, annual interest rate r, monthly payments y, and a term of N years is

$$(P_0, -y, -y, \ldots, -y),$$

where the "$-y$" term appears $12N$ times. As usual it is assumed that the interest on the loan is compounded monthly in conjunction with the payments. The present value of this cash flow stream relative to the interest rate r is

$$\begin{aligned}
PV &= P_0 - y\sum_{j=1}^{12N} \frac{1}{(1+\frac{r}{12})^j} \\
&= P_0 - y(1+\frac{r}{12})^{-12N} \sum_{j=0}^{12N-1} (1+\frac{r}{12})^j \\
&= P_0 - y(1+\frac{r}{12})^{-12N} \frac{1-(1+\frac{r}{12})^{12N}}{1-(1+\frac{r}{12})} \\
&= P_0 - y\frac{12}{r}\left(1-(1+\frac{r}{12})^{-12N})\right) \\
&= 0\ ,
\end{aligned}$$

where the final equality follows from our mortgage identity (1).

1.4. **Internal Rate of Return.** The notion of present value as the value of a cash flow stream is somewhat flawed in the sense that it requires knowledge of a prevailing interest rate r_p that is valid for the entire time horizon of the cash flow stream. One way around this difficulty is the notion of an *internal rate of return*. Note that in the example of the home mortgage we determined that the present value of the cash flow stream

$$(P_0, -y, -y, \ldots, -y)$$

was zero when the mortgage interest rate r was used as the prevailing rate of return. We could turn this on its head and ask the following question. *If this is the cash flow stream of a mortgage, then what is the interest rate being paid?* This would allow us to solve for the interest rate knowing only the initial principal P_0, the monthly payments y, and the term of the loan N years. Similarly, given a cash flow stream

$$(x_0, x_1, \ldots, x_n)$$

one can ask *what is the interest rate that gives this stream a present value of zero?* Of course, in this context, the cash flow stream should include all expenditures as well as receipts so that things can be balanced out in the end. The interest rate obtained in this way is called the internal rate of return of the cash flow stream.

Definition 1.5. *(***Internal Rate of Return***) Let $(x_0, x_1, \ldots, x_n)$ be a cash flow stream. Then the internal rate of return of this stream is the interest rate r satisfying the equation*

$$0 = x_0 + \frac{x_1}{(1+r)} + \frac{x_2}{(1+r)^2} + \cdots + \frac{x_n}{(1+r)^n} \ .$$

Equivalently, it is the number r satisfying $1/(1+r) = c$ where c satisfies the polynomial equation

$$0 = x_0 + x_1 c + x_2 c^2 + \cdots + x_n c^n \ . \tag{2}$$

Note that this definition is ambiguous since we know that the polynomial equation (2) has in general n roots. This fact can be a significant obstacle to using this notion for general cash flow streams. However, as the following result illustrates, the notion is well defined for certain types of cash flow streams.

Theorem 1.6. *(***Existence of an Internal Rate of Return***) Suppose the cash flow stream $(x_0, x_1, x_2, \ldots, x_n)$ has $x_0 < 0$ $(x_0 > 0)$ and $x_k \geq 0$ $(x_k \leq 0)$ for $k = 1, \ldots, n$ with at least one $x_k > 0$ $(x_k < 0)$. Then there is a unique positive root to the equation (2). Furthermore, if $\sum_{k=0}^{n} x_k > 0$, then the corresponding internal rate of return $r = (1/c) - 1$ is positive.*

Proof. Consider the polynomial

$$p(c) = x_0 + x_1 c + x_2 c^2 + \cdots + x_n c^n \ .$$

We have $p(0) = x_0 < 0$ while $p(c) \uparrow +\infty$ as $c \uparrow +\infty$ since $x_k \geq 0$ $k = 1, \ldots, n$ with $x_k > 0$ for at least one k. Hence the intermediate value theorem says that p must have at least one positive zero. Also, $p'(c) = x_1 + 2x_2 c + \cdots + n x_n c^{n-1}$ is strictly posiitve for $c > 0$, again since all the x_k's are non-negative with at least one stricly positive. Therefore, p is strictly increasing on the interval $[0, +\infty 0$. Consequently, p can have at most one positive root.

Next, note that if we assume that $\sum_{k=1}^{n} x_k = p(1)$ is positive, then the unique root must lie between zero and 1 in which case $r = (1/c) - 1$ is positive.

Finally, note that the case with $x_0 > 0$ follows by the same argument with the polynomial p replaced by $-p$. $\square$

If an internal rate of return can be determined, then it can be used as a measure of valuation for a cash flow stream with higher rates of return indicating a more desirable cash flow stream.

We now give an example were we compare two cash flow streams using both present value and internal rate of return. In this example we consider a scenario where we can plant cone flowers that can be later harvested and their roots, echinacea sold as a tea or herbal remedy. Cone flowers are perennials and thier root mass continues to grow each year. They also self seed very successfully. We need to make a decision as to whether the flowers should be harvested after one year or two years. For an outlay of one dollar we can expect to receive two dollars back if we harvest in one year whereas we would receive 3 dollars if we harvested after two years. With prevailing interest rate of 10% what is the most profitable harvesting strategy?

The cash flow streams associated with each of these two possible investment strategies are

1. $(-1, 2)$ (harvest in one year)
2. $(-1, 0, 3)$ (harvest in two years).

Using the present value technique of valuation we get

$$\begin{aligned} PV_1 &= -1 + \frac{2}{1.1} = 0.8182 \\ PV_2 &= -1 + \frac{3}{(1.1)^2} = 1.47 \ . \end{aligned}$$

Therefore, it seems that we should go with the three year plan as it has a greater present value.

Let us now check the internal rate of return method for valuating an investment strategy. For this we must solve the two polynomial equations

1. $-1 + 2c = 0$ (harvest in one years)
2. $-1 + 3c^2 = 0$ (harvest in two years).

The solutions are $c = 1/2$ and $c = 1/\sqrt{3}$, respectively. These roots correspond to the internal rates of return $r = (1/c) - 1 = 1$ and $r = (1/c) - 1 = \sqrt{3} - 1 = 0.7321$. Since $1 > 0.7321$, the first alternative of harvesting every year gives a better internal rate of return. This seems to contradict the present values assessment above.

Indeed, these two methods of evaluating investment opportunities can give differing and even contadictory conclusions since they are measuring two different objectives: present values versus internal rate of return. We should not be surprized by this. Nonetheless, it would be very useful if there were some way to compare them. B

To see how this might be done let us review the underlying philosophy for internal rates of return. We think of the internal rate of return as an interest rate that describes the investment strategy in terms that are similar to a classical bank savings account investment. This interest rate is a kind of prevailing interest rate for this kind of investment. As such it should be considered as ongoing even after the time horizon for the cash flow stream has past. In the context of our cone flower example, if we chose the one year harvest plan, we could replant the following year and reap the same benefits. Indeed, we could think of this as an onging strategy into the future. Similarly, we could think of the two year harvesting strategy as a strategy that cycles into the future as well. The advantage of this perspective is that now we can give both harvesting strategies the same time horizon where this horizon is equal to the least common multiple of their individual horizons.

In our cone flower example, we can consider a 2 year horizon with cash flow streams

1. $(-1, 1, 2)$ (2 harvests in two years)
2. $(-1, 0, 2.5)$ (1 harvest in two years).

In the two harvest scenario the cash flow x_1 equals the profit due to the sale of the harvest minus the cost of planting the following years harvest: $x_1 = 2 - 1$. The resulting present value from these two cash flow streams is

$$\begin{aligned} PV_1 &= -1 + \frac{1}{1.1} + \frac{2}{(1.1)^2} = 1.562 \\ PV_2 &= -1 + \frac{2.5}{(1.1)^2} = 1.47\ . \end{aligned}$$

The corresponding internal rates of return are again obtained by solving the following polynomial equations:

1. $-1 + c + 2c^2 = 0$ (2 harvest in two years)
2. $-1 + 3c^2 = 0$ (one harvest in two years).

The roots are again $c = 1/2$ and $c = 1/\sqrt{3}$ (why?), respectively. These root yield the corresponding internal rates of return $r = 1$ and $r = 0.7321$, respectively. This analysis clearly indicates that an annual harvesting strategy is prefered. Both methods of evaluation give the same advice.

It is not clear that simply putting the two investment strategies on the same time horizon will always resolve the differences between these to methods of evaluation. For example, what happens when the return on the 2 year harvesting strategy is 3.5 dollars for every dollar invested?

The debate on which of these two evaluation methods is prefered continues. In general though, the initernal rate of return approach seems prefered if the investment strategy is cyclical. Nonetheless, we will use the present value approach for most of our study. There are sound theoretical reasons for this that will become apparent as we proceed. But the most practical of these is that we wish to consider general cash flow streams, not just those for which an internal rate of return exists and can be unambiguously computed.