Math 408 Homework Set 8

(1) Locate all of the KKT points for the following problems. Can you show that these points are local solutions? Global solutions?

(a)

$$\begin{array}{ll} \text{minimize} & e^{(x_1 - x_2)} \\ \text{subject to} & e^{x_1} + e^{x_2} \le 20 \\ & 0 \le x_1 \end{array}$$

(b)

$$\begin{array}{ll} \text{minimize} & e^{(-x_1 + x_2)} \\ \text{subject to} & e^{x_1} + e^{x_2} \le 20 \\ & 0 \le x_1 \end{array}$$

(c)

$$\begin{array}{ll} \text{minimize} & x_1^2 + x_2^2 - 4x_1 - 4x_2 \\ \text{subject to} & x_1^2 \le x_2 \\ & x_1 + x_2 \le 2 \end{array}$$

(d)

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}\|x\|^2 \\ \text{subject to} & Ax = b \end{array}$$

where $b \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m \times n}$ satisfies $\text{Nul}\,(A^T) = \{0\}$.

(2) Show that the set

$$\Omega := \{x \in \mathbb{R}^2 | -x_1^3 \le x_2 \le x_1^3\}$$

is not regular at the origin. Graph the set Ω.

(3) Construct an example of a constraint region of the form (**??**) at which the MFCQ is satisfied, but the LI condition is not satisfied.

(4) Suppose $\Omega = \{x\,;\, Ax \le b,\ Ex = h\}$ where $A \in \mathbb{R}^{m\times}$, $E \in \mathbb{R}^{k \times n}$, $b \in \mathbb{R}^m$, and $h \in \mathbb{R}^k$.

(a) Given $x \in \Omega$, show that

$$T_\Omega(x) = \{d \;:\; A_{i\cdot}d \le 0 \text{ for } i \in I(x),\ Ed = 0\},$$

where $A_{i\cdot}$ denotes the ith row of the matrix A and $I(x) = \{i\ A_{i\cdot}x = b_i\}$.

(b) Given $x \in \Omega$, show that every $d \in T_\Omega(x)$ is a feasible direction for Ω at x.

(c) Note that parts (a) and (b) above show that

$$T_\Omega(x) = \bigcup_{\lambda > 0} \lambda(\Omega - x)$$

whenever Ω is a convex polyhedral set. Why?

(5) Let $C \subset \mathbb{R}^n$ be non-empty, closed and convex. For any $x \in \mathbb{R}^n$ consider the problem of finding the closest point in C to x using the 2-norm:

$$\mathcal{D} \qquad \begin{array}{ll} \text{minimize} & \frac{1}{2}\|x - z\|_2^2 \\ \text{subject to} & x \in C \end{array}.$$

Show that $\bar{z} \in C$ solves this problem if and only if

$$\langle x - \bar{z}, z - \bar{z}\rangle \le 0 \quad \text{for all } z \in C.$$

(6) Let Ω be a non-empty closed convex subset of $\mathbb{R}^n$. The geometric object *dual* to the tangent cone is called the *normal cone*:

$$N_\Omega(x) = \{z\,;\, \langle z, d\rangle \le 0,\ \text{for all } d \in T_\Omega(x)\}.$$

(a) Show that if $\bar{x}$ solves the problem $\min\{f(x) : x \in \Omega\}$ then

$$-\nabla f(\bar{x}) \in N_\Omega(\bar{x}).$$

(b) Show that

$$N_\Omega(\bar{x}) = \{z : \langle z, x - \bar{x}\rangle \leq 0, \text{ for all } x \in \Omega\}.$$

(c) Let $\bar{x} \in \Omega$. Show that $\bar{x}$ solves the problem $\min\{\frac{1}{2}\|x - y\|_2^2 : x \in \Omega\}$ for every $y \in \bar{x} + N_\Omega(\bar{x})$.

(7) Consider the functions

$$f(x) = \frac{1}{2}x^T Qx - c^T x$$

and

$$f_t(x) = \frac{1}{2}x^T Qx - c^T x + t\phi(x),$$

where $t > 0$, $Q \in \mathbb{R}^{n\times n}$ is semi-positive definite, $c \in \mathbb{R}^n$, and $\phi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is given by

$$\phi(x) = \begin{cases} -\sum_{i=1}^n \ln x_i & \text{, if } x_i > 0,\ i = 1, 2, \ldots, n, \\ +\infty & \text{, otherwise.} \end{cases}$$

(a) Show that ϕ is a convex function.
(b) Show that both f and f_t are convex functions.
(c) If Q is positive definite, show that the solution to the problem $\min f_t(x)$ always exists and is unique.
(d) Let $\{t_i\}$ be a decreasing sequence of positive real scalars with $t_i \downarrow 0$, and let x^i be the solution to the problem $\min f_{t_i}(x)$. Show that if the sequence $\{x^i\}$ has a cluster point $\bar{x}$, then $\bar{x}$ must be a solution to the problem $\min\{f(x) : 0 \leq x\}$.
Hint: Use the KKT conditions for the QP $\min\{f(x) : 0 \leq x\}$.