Math 408 Homework Set 7

(1) Show that the functions
$$f(x_1,x_2)=x_1^2+x_2^3,\quad \text{and}\quad g(x_1,x_2)=x_1^2+x_2^4$$
both have a critical point at $(x_1,x_2)=(0,0)$ and that their associated hessians are positive semi-definite. Then show that $(0,0)$ is a local (global) minimizer for g and not for f.

(2) Find the local minimizers and maximizers for the following functions if they exist:
  (a) $f(x)=x^2+\cos x$
  (b) $f(x_1,x_2)=x_1^2-4x_1+2x_2^2+7$
  (c) $f(x_1,x_2)=e^{-(x_1^2+x_2^2)}$
  (d) $f(x_1,x_2,x_3)=(2x_1-x_2)^2+(x_2-x_3)^2+(x_3-1)^2$

(3) Which of the functions in problem 2 above are convex and why?

(4) If $f:\mathbb{R}^n\to\bar{\mathbb{R}}=\mathbb{R}\cup\{+\infty\}$ is convex, show that the sets $\text{lev}_f(\alpha)=\{x\,:\,f(x)\le\alpha\}$ are convex sets for every $\alpha\in\mathbb{R}$. Let $h(x)=x^3$. Show that the sets $\text{lev}_h(\alpha)$ are convex for all α, but the function h is not itself a convex function.

(5) Show that each of the following functions is convex.
  (a) $f(x)=e^{-x}$
  (b) $f(x_1,x_2,\dots,x_n)=e^{-(x_1+x_2+\cdots+x_n)}$
  (c) $f(x)=\|x\|$

(6) If $f_i:\mathbb{R}^n\to\mathbb{R},\ i=1,2$ are convex, show that $f(x)=\max\{f_1(x),\ f_2(x)\}$ is a convex function.

(7) Let $A\in\mathbb{R}^{m\times n}$ and $b\in\mathbb{R}^m$, and suppose that $f:\mathbb{R}^m\to\mathbb{R}$ is convex. Show that $h(x)=f(Ax+b)$ is convex.

(8) Consider the functions
$$f(x)=\frac{1}{2}x^TQx-c^Tx$$
and
$$f_t(x)=\frac{1}{2}x^TQx-c^Tx+t\phi(x),$$
where $t>0$, $Q\in\mathbb{R}^{n\times n}$ is positive semi-definite, $c\in\mathbb{R}^n$, and $\phi:\mathbb{R}^n\to\mathbb{R}\cup\{+\infty\}$ is given by
$$\phi(x)=\begin{cases}-\sum_{i=1}^n\ln x_i & \text{, if } x_i>0,\ i=1,2,\dots,n,\\ +\infty & \text{, otherwise.}\end{cases}$$
  (a) Show that ϕ is a convex function.
  (b) Show that both f and f_t are convex functions.
  (c) Show that for all $t>0$ the solution to the problem $\min f_t(x)$ always exists and is unique.
  (d) Let $x(t)$ denote the unique solution to $\min f_t(x)$ for $t>0$. Show that $x(t)>0$.
  (e) Define $u(t)=t\nabla\phi(x(t))$. If there exists a sequence $t_i\downarrow 0$ and a point $(\bar{x},\bar{u})$ such that $(x(t_i),u(t_i))\to(\bar{x},\bar{u})$, show that $\bar{x}$ solves $\min_{0\le x}f(x)$.

(9) Show that each of the following functions is convex or strictly convex.
  (a) $f(x,y)=5x^2+2xy+y^2-x+2y+3$
  (b) $f(x,y)=\begin{cases}(x+2y+1)^8-\log((xy)^2), & \text{if } 0<x,\ 0<y,\\ +\infty, & \text{otherwise.}\end{cases}$
  (c) $f(x,y)=4e^{3x-y}+5e^{x^2+y^2}$
  (d) $f(x,y)=\begin{cases}x+\frac{2}{x}+2y+\frac{4}{y}, & \text{if } 0<x,\ 0<y,\\ +\infty, & \text{otherwise.}\end{cases}$

(10) Compute the global minimizers of the functions given in the previous problem if they exist.