GUIDE FOR MIDTERM EXAM 2

MATH 208A **Exam date: November 17, 2021**

The exam will have 4 questions with each question being multipart. Overall, the exam will focus on the material covered in Sections 3.1, 3.2, 3.3 and Sections 4.1, 4.2, 4.3, and 4.4 of the text. The exam will **not** test on the orthogonality material covered in class. You are allowed one handwritten 8.5 by 11 sheet of notes is allowed (2-sided is OK), and you are allowed a nonprogrammable calculator (the Texas Instruments TI-30X IIS is the official Math Dept approved calculator). A loose description of the content of each question is given below along with sample questions for the purposes of illustration and practice. The rules for the exam are listed at the end of this guide.

Question 1: This question concerns linear transformations, matrices, their properties, and the relationships between them. This includes ranges, null spaces and kernels; the Rank Plus Nullity Theorem; given properties of a linear transformation T, finding a matrix A such that $T = T_A$; examples of linear transformation or matrices having certain properties such as one-to-one, onto, specified null space, specified range, ...

Samples

(a) Let $T : \mathbb{R}^3 \mapsto \mathbb{R}^3$ be such that

$$T(e_1) = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \; T(e_2) = \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}, \; T(e_3) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Find a matrix A such that $T = T_A$.

(b) Let $T : \mathbb{R}^2 \mapsto \mathbb{R}^3$ be such that

$$T\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \; T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}.$$

(i) Write e_1 and e_2 as a linear combinations of $u^1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $u^2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

(ii) Find a matrix A such that $T = T_A$.

(c) Let $A \in \mathbb{R}^{2\times 3}$ be such that $\text{Ran}\,(A) = \text{Span}\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right]$. Give two different examples of such a matrix A.

(d) Suppose $A \in \mathbb{R}^{3\times 3}$ is such that $\text{Ran}\,(A) = \text{Span}\left[\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}\right] = S$.

(i) What is the nullity of A?

(ii) Give an example of a matrix A with $\text{Ran}\,(A) = S$ and $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \in \text{Nul}\,(A)$.

(e) Let $A, B \in \mathbb{R}^{n\times m}$ be equivalent matrices. Answer the following true or false questions.

(i) $\text{Nul}\,(A) = \text{Nul}\,(B)$: ◯ True ◯ False

(ii) $\text{Ran}\,(A) = \text{Ran}\,(B)$: ◯ True ◯ False

(f) Let $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$ be nonzero vectors and consider the matrix $A = xy^T$.

(i) If $A \in \mathbb{R}^{s\times t}$, what are s and t?

(ii) What are rank(A) and nullity(A)?

Question 2: This question concerns subspaces, their properties, bases, and their relationships to matrices such as the range and null space of a matrix as well as the column and row spaces of a matrix. This include the Rank Plus Nullity Theorem again.

(a) Compute a basis for the Ran (A), a basis for Nul (A), and a basis for Ran $\left(A^T\right)$, where

$$A = \begin{bmatrix} 1- & 1 & -3 & 0 \\ 1 & 0 & 2 & 1 \\ -1 & 3 & -5 & 2 \end{bmatrix}.$$

(b) Compute a basis for the Ran (A), a basis for Nul (A), and a basis for Ran $\left(A^T\right)$, where

$$A = \begin{bmatrix} 3 & 1 & 2 & 5 & 6 \\ 2 & 0 & 1 & 4 & 6 \\ 1 & 2 & 1 & 0 & 3 \\ 2 & 1 & 0 & 3 & -3 \end{bmatrix}.$$

Question 3: This question concerns matrix algebra including matrix multiplication. Regarding matrix multiplication, there may be questions concerning and its relationship to the composition of linear transformations, the equivalence of matrices, particularly equivalence with echelon form, matrix inversion and the properties of the inverse of a matrix.

(a) Let $A = \begin{bmatrix} 1- & 1 & -3 & 0 \\ 1 & 0 & 2 & 1 \\ -1 & 3 & -5 & 2 \end{bmatrix}$. If B is an echelon form for A, give an invertible matrix G such that $GA = B$. Also give G^{-1}.

(b) Let $A \in \mathbb{R}^{n \times k}$ and $B \in \mathbb{R}^{k \times m}$.
   (i) If $AB \in \mathbb{R}^{s \times t}$ what are s and t?
   (ii) If rank$(A) = n$ and rank$(B) = k$, what can be said about rank(AB)?
   (iii) If nullity$(A) = 0$ and nullity$(B) = 0$, what can be said about nullity(AB)?
   (iv) If $n = m$, and $k \leq n$, under what conditions is AB invertible?

(c) Give an example of a matrix A having no zero entries whose range is

$$\text{Span}\left[\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}\right]$$

and has a two dimensional null space.

Question 4: Change of bases for $\mathbb{R}^n$ and subspaces of $\mathbb{R}^n$. Coordinates of a vector in a given basis as well as the relationship between vectors and their coordinates in different bases. In particular, the computation of coordinate transformation matrices.

(a) Compute the coordinate transformation matrix for the basis

$$\mathcal{B} = \left\{\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}, \begin{pmatrix} -2 \\ 4 \\ 1 \end{pmatrix}\right\},$$

that is, compute the matrix W such that $[x]_{\mathcal{B}} = Wy$ for any vector $y \in \mathbb{R}^3$ where $[x]_{\mathcal{B}} \in \mathbb{R}^3$ is the vector containing the coordinates of y in the basis $\mathcal{B}$. Then give the coordinates of the vector $y = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ in the basis $\mathcal{B}$.

(b) Consider the following two bases for the subspace $S \subset \mathbb{R}^5$:

$$\mathcal{B}_1 = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} \right\}, \quad \mathcal{B}_2 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

Compute the coordinate transformation matrix C such that $[y]_{\mathcal{B}_2} = C[y]_{\mathcal{B}_1}$ for all $y \in S$. Then compute y and $[y]_{\mathcal{B}_2}$ for $[y]_{\mathcal{B}_1} = \begin{pmatrix} 5 \\ -1 \end{pmatrix}$. Next, compute the coordinate transformation matrix C such that $[y]_{\mathcal{B}_1} = C[y]_{\mathcal{B}_2}$ for all $y \in S$. Then compute y and $[y]_{\mathcal{B}_1}$ for $[y]_{\mathcal{B}_2} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$.

The rules for the exam are listed below.

(1) Leave your exam face up on your desk until the exam proctor tells everyone to begin.
(2) *TURN OFF YOUR PHONE!* Close all purses and backpacks and place them under your chair with your phone inside and **off**. If the proctor sees or hears a phone during the exam, you will be asked to surrender either the phone or your exam.
(3) You may use a Texas Instruments TI-30X IIS. *NO OTHER ELECTRONIC DEVICES ARE ALLOWED!*
(4) One handwritten 8.5 by 11 sheet of notes is allowed. 2-sided is OK.
(5) Only your exam, notes, calculator, and your writing implements may be on your desk during the exam.
(6) There are 4 problems. When you are told to begin the exam you must first make sure you have all of the problems. If you do not have them all, then request a new exam.
(7) Show all of your work and follow the directions provided. Partial credit will be given for partial solutions.
(8) When the exam proctor announces the end of the exam, you have two minutes to give your exam to the proctor. If the proctor does not have your exam within two minutes after the end of the exam, your exam will not be graded.
(9) At any point before or during the exam, the exam proctor may request that you change your seat. Please do so promptly.
(10) You may be asked to present a photo ID (student id or valid drivers license) at any point during the exam. If you do not have one, then you will be asked to surrender your exam.