Q 1: (a) $A = \begin{bmatrix} 1 & 3 & 1 \\ -2 & 2 & 0 \\ 1 & 2 & 1 \end{bmatrix}$

(b) (i) $e_1 = \frac{1}{2}[u^1 + u^2]$, $e_2 = \frac{1}{2}[u^2 - u^1]$

(ii) $T(e_1) = T[\frac{1}{2}(u^1+u^2)] = \frac{1}{2}[T(u^1) + T(u^2)] = \frac{1}{2}\left[\begin{pmatrix}1\\1\\1\end{pmatrix} + \begin{pmatrix}2\\1\\-1\end{pmatrix}\right]$

$= \frac{1}{2}\begin{pmatrix}3\\2\\0\end{pmatrix} = \begin{pmatrix}3/2\\1\\0\end{pmatrix}$

$T(e_2) = T(\frac{1}{2}(u^2-u^1)) = \frac{1}{2}[T(u^2) - T(u^1)] = \frac{1}{2}\left[\begin{pmatrix}2\\1\\-1\end{pmatrix} - \begin{pmatrix}1\\1\\1\end{pmatrix}\right]$

$= \frac{1}{2}\begin{pmatrix}1\\0\\-2\end{pmatrix} = \begin{pmatrix}1/2\\0\\-1\end{pmatrix}$

Therefore, $A = \begin{bmatrix} 3/2 & 1/2 \\ 1 & 0 \\ 0 & -1 \end{bmatrix}$, check: $Au^1 = \begin{pmatrix}1\\1\\1\end{pmatrix}$, $Au^2 = \begin{pmatrix}2\\1\\-1\end{pmatrix}$ ✓

(c) $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$ and $\acute{A} = \begin{bmatrix} -1 & 4 & 200 \\ -1 & 4 & 200 \end{bmatrix}$

(d) (i) rank$(A) = 2$ so nullity $= 3 - 2 = 1$

(ii) $A = \begin{bmatrix} 1 & 2 & -3 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$

Q1: (e) (i) True (ii) False since it is not always true.

A is equivalent to B if there is an invertible matrix G such that $B = GA$.

example: $A = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $G = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

so $Ran(A) \neq Ran(B)$.

(f) (i) $A \in \mathbb{R}^{n \times m}$, $s = n$, $t = m$ ex: $x = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $y = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

$x y^T = \begin{pmatrix} 1 \\ 2 \end{pmatrix} (1 \ {-1} \ 1) = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -2 & 2 \end{bmatrix}$

(ii) $Ran(xy^T) = span[x]$ if $y \neq 0$, else $Ran(xy^T) = \{0\}$

$rank(xy^T) = 1$ if $y \neq 0$ and $x \neq 0$

otherwise $rank(xy^T) = 0$ if either $x = 0$ or $y = 0$.

$nullity(xy^T) = m - 1$ if neither $x = 0$ or $y = 0$

otherwise $nullity(xy^T) = m$

Q2: (a)

$$\begin{matrix} 1 & -1 & -3 & 0 \\ 1 & 0 & 2 & 1 \\ -1 & 3 & -5 & 2 \end{matrix}$$

$$\begin{matrix} 1 & 0 & 2 & 1 \\ 0 & -1 & -5 & -1 \\ 0 & 3 & -3 & 3 \end{matrix}$$

$$\begin{matrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 5 & 1 \\ 0 & 0 & -18 & 0 \end{matrix}$$

$$\begin{matrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{matrix}$$

Basis for Ran(A) $= \left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix} \begin{pmatrix} -3 \\ 2 \\ -5 \end{pmatrix} \right\}$

Basis for Nul(A) $= \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \end{pmatrix} \right\}$

Basis for Ran(A^T) $= \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$

(b)

$$\begin{matrix} 3 & 1 & 2 & 5 & 6 \\ 2 & 0 & 1 & 4 & 6 \\ 1 & 2 & 1 & 0 & 3 \\ 2 & 1 & 0 & 3 & -3 \end{matrix}$$

$$\begin{matrix} 1 & 2 & 1 & 0 & 3 \\ 0 & -5 & -1 & 5 & -3 \\ 0 & -4 & -1 & 4 & 0 \\ 0 & -3 & -2 & 3 & -9 \end{matrix}$$

$$\begin{matrix} 1 & 2 & 1 & 0 & 3 \\ 0 & 1 & 0 & -1 & 3 \\ 0 & 0 & -1 & 0 & 12 \\ 0 & 0 & -2 & 0 & 0 \end{matrix}$$

$$\begin{matrix} 1 & 0 & 1 & 2 & -3 \\ 0 & 1 & 0 & -1 & 3 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 0 & 12 \end{matrix}$$

$$\begin{matrix} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{matrix}$$

Basis

Ran(A), $\left\{ \begin{pmatrix} 3 \\ 2 \\ 1 \\ 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 6 \\ 6 \\ 3 \\ -3 \end{pmatrix} \right\}$

Nul(A), : $\left\{ \begin{pmatrix} 2 \\ -1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \right\}$

Ran(A^T) $= \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 2 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

Q3: (a)

$$\left[\begin{array}{cccc|ccc} 1 & -1 & -3 & 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 1 & 0 & 1 & 0 \\ -1 & 3 & -5 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cccc|ccc} 1 & 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & -1 & -5 & -1 & 1 & -1 & 0 \\ 0 & 3 & -3 & 3 & 0 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{cccc|ccc} 1 & 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 5 & 1 & -1 & 1 & 0 \\ 0 & 0 & -18 & 0 & 3 & -2 & 1 \end{array}\right] \Big\} G$$

$$G = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 1 & 0 \\ 3 & -3 & 1 \end{bmatrix}$$

Check

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & -3 & 0 \\ 1 & 0 & 2 & 1 \\ -1 & 3 & -5 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 5 & 1 \\ 0 & 0 & -18 & 0 \end{bmatrix} \checkmark$$

Compute G^{-1}

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 3 & -2 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 3 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 3 & 1 \end{array}\right]$$

Check $\begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & 0 \\ -1 & 3 & 1 \end{pmatrix}\begin{bmatrix} 0 & 1 & 0 \\ -1 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\checkmark$

Q3: (b) (i) $s = n$, $m = t$

(ii) $\operatorname{rank}(AB) = n$. Reason: $\operatorname{rank}(A) = n \iff \operatorname{Ran}(A) = \mathbb{R}^n$

$\operatorname{rank}(B) = k \iff \operatorname{Ran}(B) = \mathbb{R}^k$

Let $y \in \mathbb{R}^n$. $\operatorname{Ran}(A) = \mathbb{R}^n \Rightarrow \exists\, x$ such that $y = Ax$. $\operatorname{Ran}(B) = \mathbb{R}^k \Rightarrow \exists\, z$ s.t. $x = Bz$.

Therefore $ABz = Ax = y$ so $\operatorname{Ran}(AB) = \mathbb{R}^n$ so $\operatorname{rank}(AB) = n$

(iii) $\operatorname{nullity}(AB) = 0$: Reason: $\operatorname{nullity}(A) = 0 \iff \operatorname{Nul}(A) = \{0\}$

$\operatorname{nullity}(B) = 0 \iff \operatorname{Nul}(B) = \{0\}$

so $ABz = 0 \iff Bz = 0 \Rightarrow z = 0$

Therefore $\operatorname{Nul}(AB) = \{0\}$ so $\operatorname{nullity}(AB) = 0$

(iv) We must have $k = n$ and both A and B are invertible.

Reason: If $k < n$ then $\operatorname{nullity}(B) \geq n - k > 0$ so $\operatorname{Nul}(B) \neq \{0\} \Rightarrow \operatorname{Nul}(AB) \neq 0$ so $k = n$.

If B is not invertible, then $\operatorname{Nul}(B) \neq \{0\}$ so $\operatorname{Nul}(AB) \supset \operatorname{Nul}(B) \neq \{0\}$ so AB not invertible. So B must be invertible.

If A not invertible $\Rightarrow \exists\, x \neq 0$ s.t. $Ax = 0$. Set $y = B^{-1}x$ so $By = x \Rightarrow ABy = Ax = 0$ so AB not invertible so A^{-1} exist.

Q3: (C) $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$, $\operatorname{Ran} A = \operatorname{Span}\left[\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right]$

$\operatorname{Nul} A = \operatorname{Span}\left[\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \right]$

<u>Motivation</u>: Rank + Nullity Theorem $\Rightarrow A \in \mathbb{R}^{3 \times 4}$

$$\left[\begin{array}{cc|ccc} 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{array}\right] \quad \left[\begin{array}{ccc|ccc} 1/2 & 1/2 & 0 & 1 & 0 & 0 \\ 1/2 & -1/2 & 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|ccc} 1 & 1 & 1 & 0 & 0 \\ 0 & -2 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{array}\right] \quad \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 2 & 0 & 0 \\ 1 & -1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 1 & 2 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|ccc} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1/2 & -1/2 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{array}\right] \quad \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 2 & 0 & 0 \\ 0 & -2 & 0 & -2 & 2 & 0 \\ 0 & 1 & 1 & 2 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|ccc} 1 & 0 & 1/2 & 1/2 & 0 \\ 0 & 1 & 1/2 & -1/2 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{array}\right] \Big\} G \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right] \Big\} G^{-1}$$

Check

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & -1/2 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \checkmark$$

$$GA = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A = G^{-1} \underbrace{\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\operatorname{Nul}(A) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \right\}$$

Q4: (a) $u = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 0 & 4 \\ 2 & 4 & 1 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 1 & 0 & 0 \\ -1 & 0 & 4 & 0 & 1 & 0 \\ 2 & 4 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -4 & 0 & -1 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 4 & 9 & 0 & 2 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -4 & 0 & -1 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & -4 & -2 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -16 & -9 & 4 \\ 0 & 1 & 0 & 9 & 5 & -2 \\ 0 & 0 & 1 & -4 & -2 & 1 \end{array}\right] W$$

check

$$\begin{bmatrix} 1 & 1 & -2 \\ -1 & 0 & 4 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} -16 & -9 & 4 \\ 9 & 5 & -2 \\ -4 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \checkmark$$

$$Wy = \begin{bmatrix} -16 & -9 & 4 \\ 9 & 5 & -2 \\ -4 & -2 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$= \begin{bmatrix} -16 & -18 & +12 \\ 9 & +10 & -6 \\ -4 & -4 & +3 \end{bmatrix} = \begin{pmatrix} -22 \\ 13 \\ -5 \end{pmatrix}$$

Q4: b) We must compute $[u^1]_{\mathcal{B}_2}$ and $[u^2]_{\mathcal{B}_2}$

where $u^1 = \begin{pmatrix}1\\1\\1\\1\\1\end{pmatrix}$, $u^2 = \begin{pmatrix}1\\-1\\1\\-1\\1\end{pmatrix}$

$$\left[\begin{array}{cc|cc} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 1 & 0 & 1 & 1 \\ \hline 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow [u^1]_{\mathcal{B}_2} = \begin{pmatrix}1\\1\end{pmatrix}$ $[u^2]_{\mathcal{B}_2} = \begin{pmatrix}1\\-1\end{pmatrix}$ $\Rightarrow C = \begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}$

$[y]_{\mathcal{B}_1} = \begin{pmatrix}5\\1\end{pmatrix} \Rightarrow y = \begin{bmatrix}1 & 1\\1 & -1\\1 & 1\\1 & -1\\1 & 1\end{bmatrix}\begin{pmatrix}5\\1\end{pmatrix} = \begin{pmatrix}6\\4\\6\\4\\6\end{pmatrix}$

$[y]_{\mathcal{B}_2} = C[y]_{\mathcal{B}_1} = \begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\begin{pmatrix}5\\1\end{pmatrix} = \begin{bmatrix}6\\4\end{bmatrix}$

check $\begin{pmatrix}1 & 0\\0 & 1\\1 & 0\\0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}6\\4\end{pmatrix} = \begin{pmatrix}6\\4\\6\\4\\6\end{pmatrix}$ ✓

$[y]_{\mathcal{B}_1} = C^{-1}\begin{pmatrix}-1\\2\end{pmatrix} = \frac{1}{2}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\begin{pmatrix}-1\\2\end{pmatrix} = \begin{pmatrix}1/2\\-3/2\end{pmatrix}$

$[y]_{\mathcal{B}_2} = \begin{bmatrix}-1\\2\end{bmatrix}$

$\Rightarrow \begin{bmatrix}1 & 0\\0 & 1\\1 & 0\\0 & 1\\1 & 0\end{bmatrix}\begin{pmatrix}-1\\2\end{pmatrix} = \begin{pmatrix}-1\\2\\-1\\2\\-1\end{pmatrix}$

Compute C^{-1}

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \\ \hline 1 & 1 & 1 & 0 \\ 0 & -2 & -1 & 1 \\ \hline 1 & 0 & 1/2 & 1/2 \\ 0 & 1 & 1/2 & -1/2 \end{array}\right]$$

check $\begin{bmatrix}1/2 & 1/2\\1/2 & -1/2\end{bmatrix}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ ✓

check $\begin{pmatrix}1 & 1\\1 & -1\\1 & 1\\1 & -1\\1 & 1\end{pmatrix}\begin{pmatrix}1/2\\-3/2\end{pmatrix} = \begin{pmatrix}-1\\2\\-1\\2\\-1\end{pmatrix}$