FINAL EXAM GUIDE SOLUTIONS

MATH 208A **Exam Date: December 13, 2021**

The exam will have 6 questions with each question being multipart. As with the midterm exam, the questions are not equally weighted. The exam covers all of the key concepts introduced in this course. Specifically, it covers the material in Sections 1.1, 1.2, 2.1, 2.1, 2.3, 3.1, 3.2, 3.3, 4.1, 4.2, 4.3, 4.4, 5.1, 5.2, 6.1, and 6.2 of the text. In particular, this means that the exam does **not** test on the orthogonality material covered in class. You are allowed one handwritten 8.5 by 11 sheet of notes is allowed (2-sided is OK), and you are allowed a nonprogrammable calculator (the Texas Instruments TI-30X IIS is the official Math Dept approved calculator). A loose description of the content of each question is given below along with sample questions for the purposes of illustration and practice. The rules for the exam are listed at the end of this guide.

Question 1: Samples

Our general tool for solving systems of the form $Ax = b$, where $A \in \mathbb{R}^{n\times m}$ and $b \in \mathbb{R}^n$ is to reduce the augmented system $[A|b]$ to echelon form using the elementary row operations. Since the elementary row operations are equivalent to multiplying the augmented matrix $[A|b]$ on the left by an invertible elementary matrix G_1, we find after the first elementary row operation that we have the augmented matrix $G_1[A|b] = [G_1A|G_1b]$ and after the second, we have $G_2G_1[A|b] = G_2[G_1A|G_1b] = [G_2G_1A|G_2G_1b]$, and after k elementary row operations we have

$$G[A|b] = G_k \cdots G_2G_1[A|b] = [G_k \cdots G_2G_1A|G_k \cdots G_2G_1b] = [GA|Gb],$$

where $G = G_k \cdots G_2G_1$. Now suppose we apply the *same* elementary row operations to the augmented matrix $[A|I]$. The we would get

$$G[A|I] = G_k \cdots G_2G_1[A|I] = [G_k \cdots G_2G_1A|G_k \cdots G_2G_1] = [GA|G],$$

that is, we would recover the matrix G that puts A into echelon form.

(a) Consider the linear system

$$\begin{array}{rcrcrcrcrcr} x_1 & + & x_2 & + & 3x_3 & + & 2x_4 & + & 2x_5 & = & 2 \\ 2x_1 & + & x_2 & + & 5x_3 & + & x_4 & + & 2x_5 & = & -2 \\ 4x_1 & + & 3x_2 & + & 11x_3 & + & 3x_4 & + & 4x_5 & = & 2 \\ 3x_1 & + & 2x_2 & + & 8x_3 & + & 3x_4 & + & 4x_5 & = & 0 \end{array}.$$

(i) Write the augmented matrix A associated with this system.
(ii) Compute an echelon form for this matrix.
(iii) Compute a reduced echelon form for this augmented matrix.

Solution:

1	1	3	2	2	1	0	0	0	
2	1	5	1	2	0	1	0	0	
4	3	11	3	4	0	0	1	0	
3	2	8	3	4	0	0	0	1	
1	0	2	−1	0	−1	1	0	0	
0	1	1	1	0	0	−2	1	0	echelon
0	0	0	2	2	2	1	−1	0	form
0	0	0	0	0	−1	−1	0	1	
1	0	2	0	1	0	3/2	−1/2	0	reduced
0	1	1	0	−1	−1	−5/2	3/2	0	echelon
0	0	0	1	1	1	1/2	−1/2	0	form
0	0	0	0	0	1	1	0	−1	

Then $G = \frac{1}{2}\begin{bmatrix} 0 & 3/2 & -1/2 & 0 \\ -1 & -5/2 & 3/2 & 0 \\ 1 & 1/2 & -1/2 & 0 \\ 1 & 1 & 0 & -1 \end{bmatrix}$ with $GA = B$ where

$$A = \begin{bmatrix} 1 & 1 & 3 & 2 & 2 \\ 2 & 1 & 5 & 1 & 2 \\ 4 & 3 & 11 & 3 & 4 \\ 3 & 2 & 8 & 3 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(iv) Write a description of the set of solutions to this system in vector form.

(v) If B is the reduced echelon form for the augmented matrix, compute an invertible matrix $G \in \mathbb{R}^{4\times 4}$ such that $GA = B$.

(vi) Let $c_0 = (2,\ -2,\ 2,\ 0)^T$ be the vector given by the right-hand side of the linear system given above. Compute the vector Gc_0 and explain the relationship of this vector to the solution set to this linear system.

Solution: Write $\bar{x} = G\begin{pmatrix} 2 \\ -2 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 \\ 6 \\ 0 \\ 0 \end{pmatrix}$. Then the solution set is given by solving the augmented system $[B|\bar{x}]$. or

$$\left|\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & 1 & -4 \\ 0 & 1 & 1 & 0 & -1 & 6 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right|$$

giving the solution set

$$\left\{ \begin{pmatrix} -4 \\ 6 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} -2 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} -1 \\ 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} \ : \ s, t \in \mathbb{R} \right\}.$$

(vii) Let $c_1 = (1,\ 1,\ 1,\ 2)^T$. Use the matrix G to describe the set of solutions to the linear system obtained from the system above by replacing the right-hand side vector c_0 by the the vector c_1.

Solution: Write $G\begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}$ so the associated augmented system is

$$\left|\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 & -1 & -2 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right|$$

yielding the solutions set

$$\left\{ \begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} -1 \\ 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} \; : \; s,t \in \mathbb{R} \right\}.$$

(viii) Suppose the right-hand side vector of the linear system given above is evolving in time. You are given 1000 right-hand side vectors for the linear system where $c_i \in \mathbb{R}^4$ the right-hand side at time point t_i for $i = 1, 2, \ldots, 1000$. Describe a process for solving all of these linear systems by taking the product of two matrices (Hint: One of these matrices is in $\mathbb{R}^{4\times 4}$ and the other is in $\mathbb{R}^{4\times 10,000}$).

Solution: Write $Gc_i = \begin{pmatrix} x_{i1} \\ x_{i2} \\ x_{i3} \\ x_{i4} \end{pmatrix}$, $i = 1, 2, \ldots, 1000$ yielding the associated augmented system

$$B = \left|\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & 1 & x_{i1} \\ 0 & 1 & 1 & 0 & -1 & x_{i2} \\ 0 & 0 & 0 & 1 & 1 & x_{i3} \\ 0 & 0 & 0 & 0 & 0 & x_{i4} \end{array}\right|.$$

Hence, for every $j \in \{1, 2, \ldots, 1000\}$ for which $x_{i4} \neq 0$, the system $Ax = c_i$ is inconsistent (Why?), while for every $j \in \{1, 2, \ldots, 1000\}$ for which $x_{i4} = 0$, we obtain the solution set

$$\left\{ \begin{pmatrix} x_{i1} \\ x_{i2} \\ 0 \\ x_{i3} \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} -1 \\ 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} \; : \; s,t \in \mathbb{R} \right\}.$$

(b) Answer questions (i)-(v) of (a) for the linear system

$$\begin{array}{rcrcrcrcrcr} x_1 & + & x_2 & + & 2x_3 & + & x_4 & + & x_5 & = & 2 \\ x_1 & + & 3x_2 & - & 2x_3 & - & 3x_4 & + & 5x_5 & = & -2 \\ 2x_1 & + & 5x_2 & - & 2x_3 & - & 2x_4 & + & 6x_5 & = & 2 \end{array}.$$

Question 2: This question concerns the concept of a subspace, linear independence, bases, dimension. Of particular importance are the subspaces $\text{Ran}\,(A) = \text{col}(A)$, $\text{Nul}\,(A)$, $\text{Ran}\left(A^T\right) = \text{row}(A)$, and $\text{Nul}\left(A^T\right)$, how to compute bases for each using (reduced) echelon form, and the Rank Plus Nullity Theorem.

(a) For the following two matrices compute bases for the subspaces $\text{Ran}\,(A) = \text{col}(A)$, $\text{Nul}\,(A)$, $\text{Ran}\left(A^T\right) = \text{row}(A)$, and $\text{Nul}\left(A^T\right)$ and give the dimension of each of these subspaces.

(i) $A = \begin{bmatrix} 1 & 2 & 3 & 2 & 2 \\ 2 & 1 & 5 & 1 & 2 \\ 4 & 3 & 11 & 3 & 4 \\ 3 & 2 & 8 & 3 & 4 \end{bmatrix}$

Solution: Compute

$$[GA|G] = \left|\begin{array}{ccccc|cccc} 1 & 0 & 0 & 0 & 3 & 2 & 13/2 & -7/2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & -1 & -5/2 & 3/2 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1/2 & -1/2 & 1 \end{array}\right| \quad G = \frac{1}{2}\begin{bmatrix} 4 & 13 & -7 & 0 \\ 2 & 2 & 0 & -2 \\ -2 & -5 & 3 & 0 \\ 0 & -1 & -1 & 2 \end{bmatrix}.$$

Therefore,

$$\left\{\begin{pmatrix}1\\2\\4\\3\end{pmatrix}, \begin{pmatrix}2\\1\\3\\2\end{pmatrix}, \begin{pmatrix}3\\5\\11\\8\end{pmatrix}, \begin{pmatrix}2\\1\\3\\3\end{pmatrix}\right\} \quad \text{is a basis for Ran}\,(A) \text{ (or just use the standard basis as } A \text{ is onto),}$$

$$\left\{\begin{pmatrix}-3\\0\\1\\-1\\1\end{pmatrix}\right\} \quad \text{is a basis for Nul}\,(A),$$

$$\left\{\begin{pmatrix}1\\0\\0\\0\\3\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\\-1\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\\1\end{pmatrix}\right\} \quad \text{is a basis for Ran}\left(A^T\right)\text{, and}$$

A is onto so $\mathrm{Nul}\left(A^T\right) = \{0\}$

(ii) $A = \begin{bmatrix} 1 & 1 & 2 & 1 & 1 \\ 1 & 3 & -2 & -3 & 5 \\ 2 & 5 & -2 & -2 & 6 \end{bmatrix}$

(b) For the matrices A in part (a) chose a basis for the row space of A from the rows of A.

(c) Answer the following questions for the matrices $A = [S\ T]$ and $B = A^T$ where $S \in \mathbb{R}^{n\times n}$ is invertible and $T \in \mathbb{R}^{n\times t}$ with $t \geq 1$.

Solution: To answer the following questions observe that $\mathrm{rank}(A^T) = \mathrm{rank}(A) = \mathrm{rank}(S) = n$ and $\mathrm{nullity}(A) = (n+t) - \mathrm{rank}(A) = t$.

(i) Is A onto? Is A one to one? What is the rank of A? What is the nullity of A?

Solution: A is onto since $\mathrm{rank}(A) = n$. A is not one to one since $\mathrm{nullity}(A) = t \geq 1$.

(ii) Is B onto? Is B one to one? What is the rank of B? What is the nullity of B?

Solution: B is not onto since $\mathrm{rank}(B) = \mathrm{rank}(A^T) = n < n+t$. B is one to one since $\mathrm{nullity}(B) = n - \mathrm{rank}(B) = 0$.

(iii) Set $L = \begin{bmatrix} S & T \\ 0 & K \end{bmatrix}$ where $K \in \mathbb{R}^{t\times t}$. Under what conditions is L both onto and one to one?

Solution: K needs to be invertible since then L is invertible because $\det(L) = \det(S)\det(K) \neq 0$.

Solution: For (d)-(j) see midterm guide 2.

(d) Let $A \in \mathbb{R}^{2\times 3}$ be such that $\operatorname{Ran}(A) = \operatorname{Span}\left[\begin{pmatrix}1\\1\end{pmatrix}\right]$. Give two different examples of such a matrix A.

(e) Suppose $A \in \mathbb{R}^{3\times 3}$ is such that $\operatorname{Ran}(A) = \operatorname{Span}\left[\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}2\\-1\\0\end{pmatrix}\right] = S$.

(i) What is the nullity of A?

(ii) Give an example of a matrix A with $\operatorname{Ran}(A) = S$ and $\begin{pmatrix}1\\1\\1\end{pmatrix} \in \operatorname{Nul}(A)$.

(f) Let $A, B \in \mathbb{R}^{n\times m}$ be equivalent matrices. Answer the following true or false questions.

(i) $\operatorname{Nul}(A) = \operatorname{Nul}(B)$: ○ True ○ False

(ii) $\operatorname{Ran}(A) = \operatorname{Ran}(B)$: ○ True ○ False

(g) Let $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$ be nonzero vectors and consider the matrix $A = xy^T$.

(i) If $A \in \mathbb{R}^{s\times t}$, what are s and t?

(ii) What are $\operatorname{rank}(A)$ and $\operatorname{nullity}(A)$?

(h) Compute a basis $\operatorname{Ran}(A)$, $\operatorname{Nul}(A)$, $\operatorname{Ran}\left(A^T\right)$, and $\operatorname{Nul}\left(A^T\right)$ where

$$A = \begin{bmatrix} 1 & -1 & -3 & 0 \\ 1 & 0 & 2 & 1 \\ -1 & 3 & -5 & 2 \end{bmatrix}.$$

(i) Compute a basis $\operatorname{Ran}(A)$, $\operatorname{Nul}(A)$, $\operatorname{Ran}\left(A^T\right)$, and $\operatorname{Nul}\left(A^T\right)$ where

$$A = \begin{bmatrix} 3 & 1 & 2 & 5 & 6 \\ 2 & 0 & 1 & 4 & 6 \\ 1 & 2 & 1 & 0 & 3 \\ 2 & 1 & 0 & 3 & -3 \end{bmatrix}.$$

(j) Let $A \in \mathbb{R}^{n\times k}$ and $B \in \mathbb{R}^{k\times m}$.

(i) If $AB \in \mathbb{R}^{s\times t}$ what are s and t?

(ii) If $\operatorname{rank}(A) = n$ and $\operatorname{rank}(B) = k$, what can be said about $\operatorname{rank}(AB)$?

(iii) If $\operatorname{nullity}(A) = 0$ and $\operatorname{nullity}(B) = 0$, what can be said about $\operatorname{nullity}(AB)$?

(iv) If $n = m$, and $k \le n$, under what conditions is AB invertible?

Question 3: This question concerns linear transformations and their matrix representation, matrix algebra including the relationship between matrix multiplication and the composition of linear transformations, and the inverse of a matrix including its computation, and properties, their use in solving linear systems including the relationship to echelon form and the equivalence of linear systems.

(a) Let $T : \mathbb{R}^m \mapsto \mathbb{R}^n$ be a linear tranformation. Let $\{b_1, b_2, \ldots, b_m\}$ be a basis for $\mathbb{R}^m$ and suppose that $T(b_i) = y_i$ for $y_i \in \mathbb{R}^n$, $i = 1, 2, \ldots, m$. In terms of the matrices $B = [b_1\ b_2\ \ldots\ b_m]$ and $Y = [y_1\ y_2\ \ldots y_m]$ derive an expression for the matrix A such that $T = T_A$, i.e. $T(x) = Ax$.

Solution: If $T = T_A$, then $AB = Y$ so $A = YB^{-1}$ since B is invertible.

(b) Let $T : \mathbb{R}^3 \mapsto \mathbb{R}^3$ be such that

$$T(e_1) = \begin{pmatrix}1\\-2\\1\end{pmatrix},\ T(e_2) = \begin{pmatrix}3\\2\\2\end{pmatrix},\ T(e_3) = \begin{pmatrix}1\\0\\1\end{pmatrix}.$$

Find a matrix A such that $T = T_A$. (Hint: Use the result from part (a) above.)

Solution: $A = \begin{bmatrix} 1 & 3 & 1 \\ -2 & 2 & 0 \\ 1 & 2 & 1 \end{bmatrix}$.

(c) Let $T : \mathbb{R}^3 \mapsto \mathbb{R}^2$ be such that

$$T\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \; T\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \; T\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

Find a matrix A such that $T = T_A$.

Solution: Since the columns of $B = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 1 & -1 & -1 \end{bmatrix}$ are linearly independent, we know from the observation in part (a) that $A = YB^{-1}$ where $Y = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 2 \end{bmatrix}$. The usual computation shows that $B^{-1} = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 2 & -1 \\ 2 & -3 & 1 \end{bmatrix}$ so

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ -1 & 2 & -1 \\ 2 & -3 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 6 & -2 \\ 4 & -5 & 2 \end{bmatrix}.$$

(d) Let $A = \begin{bmatrix} 1 & -1 & -3 & 0 \\ 1 & 0 & 2 & 1 \\ -1 & 3 & -5 & 2 \end{bmatrix}$. If B is an echelon form for A, give an invertible matrix G such that $GA = B$. Also give G^{-1}.

Solution:

$$G = \frac{1}{2}\begin{bmatrix} 2 & -2 & -2 \\ -3 & 8 & 5 \\ -1 & 2 & 1 \end{bmatrix} \quad \text{and} \quad G^{-1} = \begin{bmatrix} 1 & 1 & -3 \\ 1 & 0 & 2 \\ -1 & 1 & -5 \end{bmatrix}.$$

(e) Give an example of a matrix A having no zero entries whose range is

$$\text{Span}\left[\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}\right]$$

and has a two dimensional null space.

(f) Let $u \in \mathbb{R}^n$ and $v \in \mathbb{R}^m$. Give an example of a matrix A such that $\text{Ran}\,(A) = \text{Span}\,[u]$ and $\text{Nul}\,(A) = \{x \in \mathbb{R}^m \,:\, v^T x = 0\}$. What are the rank and nullity of A?

Solution: $A = uv^T$. $\text{rank}(A) = 1$ since $\text{Ran}\,(A) = \text{Span}\,[u]$ and $\text{nullity}(A) = m - \text{rank}(A) = m - 1$

Question 4: This question concerns basis representations for subspaces using bases, the coordinate representation of vectors in a given basis, the computation of coordinate transformation matrices allowing the transformation of the coordinates of a vectors between bases, and the change of bases matrices for subspaces of $\mathbb{R}^n$ and their computation.

(a) Compute the coordinate transformation matrix for the basis

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}, \begin{pmatrix} -2 \\ 4 \\ 1 \end{pmatrix} \right\},$$

that is, compute the matrix W such that $[x]_{\mathcal{B}} = Wy$ for any vector $y \in \mathbb{R}^3$ where $[x]_{\mathcal{B}} \in \mathbb{R}^3$ is the vector containing the coordinates of y in the basis $\mathcal{B}$. Then give the coordinates of the vector $y = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ in the basis $\mathcal{B}$.

Solution: $W = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 0 & 4 \\ 2 & 4 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -16 & -9 & 4 \\ 9 & 5 & -2 \\ -4 & -2 & 1 \end{bmatrix}$ so
$[y]_{\mathcal{B}} = Wy = \begin{bmatrix} -16 & -9 & 4 \\ 9 & 5 & -2 \\ -4 & -2 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -22 \\ 13 \\ -5 \end{pmatrix}.$

(b) Consider the following two bases for the subspace $S \subset \mathbb{R}^5$:

$$\mathcal{B}_1 = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} \right\}, \quad \mathcal{B}_2 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

Compute the coordinate transformation matrix C such that $[y]_{\mathcal{B}_2} = C[y]_{\mathcal{B}_1}$ for all $y \in S$. Then compute y and $[y]_{\mathcal{B}_2}$ for $[y]_{\mathcal{B}_1} = \begin{pmatrix} 5 \\ -1 \end{pmatrix}$. Next, compute the coordinate transformation matrix C such that $[y]_{\mathcal{B}_1} = C[y]_{\mathcal{B}_2}$ for all $y \in S$. Then compute y and $[y]_{\mathcal{B}_1}$ for $[y]_{\mathcal{B}_2} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$.

Solution: $C = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ and $C^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ so $[y]_{\mathcal{B}_1} = C[y]_{\mathcal{B}_2}$ for all $y \in S$, and $[y]_{\mathcal{B}_2} = C^{-1}[y]_{\mathcal{B}_1}$ for all $y \in S$.

(c) Consider the following two bases for the subspace $S \subset \mathbb{R}^5$:

$$\mathcal{B}_1 = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -2 \\ 1 \end{pmatrix} \right\}, \quad \mathcal{B}_2 = \left\{ \begin{pmatrix} -1 \\ 5 \\ 8 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \\ -5 \\ 0 \end{pmatrix} \right\}.$$

(i) Compute the coordinate transformation matrix C such that $[y]_{\mathcal{B}_1} = C[y]_{\mathcal{B}_2}$ for all $y \in S$.

(ii) Then compute y and $[y]_{\mathcal{B}_1}$ for $[y]_{\mathcal{B}_2} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$.

Solution: $C = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$ and $C^{-1} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ so $[y]_{\mathcal{B}_1} = C[y]_{\mathcal{B}_2}$ for all $y \in S$, and $[y]_{\mathcal{B}_2} = C^{-1}[y]_{\mathcal{B}_1}$ for all $y \in S$.

(d) Do the vectors the sets $\mathcal{B}_1$ and $\mathcal{B}_2$ (given below) span the same subspace of $\mathbb{R}^5$?

$$\mathcal{B}_1 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\} \quad \mathcal{B}_2 = \left\{ \begin{pmatrix} 1 \\ 1 \\ 3 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 5 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ 3 \\ 11 \\ 3 \\ 4 \end{pmatrix} \right\}.$$

If they do span the same subspace, compute the coordinate transformation matrix C for which $[y]_{\mathcal{B}_1} = C[y]_{\mathcal{B}_2}$ for all $y \in \mathbb{R}^5$. (Hint: Start by trying to compute C and see what happens.)

Solution: $C = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 1 & 3 \\ 2 & 1 & 3 \end{bmatrix}$

Question 5: This question concerns the determinant of a matrix including its properties, computation, use, and relationship to linear systems.

(a) Use the properties of the deteminant to compute the determinant of the following matrices.

(i) $A = \begin{bmatrix} 2 & 1 & 2 & -7 & 11 & 24 \\ 1 & 2 & 1 & 1 & -17 & 2 \\ 0 & 0 & 1 & -1 & 6 & 8 \\ 0 & 0 & 1 & 4 & -2 & -5 \\ 0 & 0 & 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$

Solution: $|A| = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} \begin{vmatrix} 1 & -1 \\ 1 & 4 \end{vmatrix} \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = 3 \cdot 5 \cdot 1 = 15.$

(ii) $B = \begin{bmatrix} 1 & 3 & 2 & 1 \\ 0 & 2 & 4 & 1 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 5 & 6 \end{bmatrix}$

Solution: $|B| = 1 \cdot 2 \cdot (-5) \cdot 6 = -60.$

(iii) $C = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 4 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$

Solution: Reduce

$$\begin{array}{rrrr} 1 & 0 & 0 & 1 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 4 & 0 \\ -1 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 1 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{array}.$$

so $|C| = 1 \cdot 2 \cdot 2 \cdot 2 = 8.$

(b) Let $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ where $A \in \mathbb{R}^{s \times s}$ is invertible, $B \in \mathbb{R}^{s \times t}$, $C \in \mathbb{R}^{t \times s}$, and $D \in \mathbb{R}^{t \times t}$.

(i) Compute the matrix product $\begin{bmatrix} I_s & 0 \\ -CA^{-1} & I_t \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix}$.

Solution: $\begin{bmatrix} I_s & 0 \\ -CA^{-1} & I_t \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & D - CA^{-1}B \end{bmatrix}$.

(ii) Use the computation above to show that $\det(M) = \det(A)\det(D - CA^{-1}B)$.

Solution:

$$\begin{aligned}
\det(M) &= \det\left(\begin{bmatrix} I_s & 0 \\ -CA^{-1} & I_t \end{bmatrix}\right)\det(M) \\
&= \det\left(\begin{bmatrix} I_s & 0 \\ -CA^{-1} & I_t \end{bmatrix}\right)\det\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \\
&= \det\left(\begin{bmatrix} I_s & 0 \\ -CA^{-1} & I_t \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \\
&= \det\left(\begin{bmatrix} A & B \\ 0 & D - CA^{-1}B \end{bmatrix}\right) \\
&= \det(A)\det(D - CA^{-1}B).
\end{aligned}$$

(c) Let the columns of the matrix $B = [b_1\ b_2, \ldots, b_m] \in \mathbb{R}^{m\times m}$ form a basis for $\mathbb{R}^m$ and suppose that $T : \mathbb{R}^m \mapsto \mathbb{R}^m$ is a linear transformation such that $T(b_i) = y_i \in \mathbb{R}^m,\ i = 1, 2, \ldots, m$. If $A \in \mathbb{R}^{m\times m}$ is such that $T = T_A$ ($T_A(x) = Ax$ for all $x \in \mathbb{R}^m$), show that $\det(A) = \frac{\det(Y)}{\det(B)}$, where $Y = [y_1,\ y_2, \ldots, y_m]$. (Hint: Review the solution to question 3 part (a).)

Solution: $T_A(b_i) = Ab_i$, so $AB = [Ab_1,\ Ab_2,\ \ldots,\ Ab_m] = [[y_1,\ y_2, \ldots, y_m] = Y$ which gives $A = YB^{-1}$. Therefore, $\det(A) = \det(YB^{-1}) = \det(Y)\det(B^{-1}) = \det(Y)\det(B)^{-1}$.

Question 6: This question concerns the eigenvectors and eigenvalues of an $n \times n$ real matrix. This includes the eigenspace associated with an eigenvalue, the characteristic polynomial, the computation of eigenvalues and eigenvectors, the algebraic and geometric multiplicity of eigenvalues, and diagonalizable matrices and their properties.

(a) Compute the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} -1 & -1 & 0 \\ 3 & 1 & -1 \\ -5 & -1 & 2 \end{bmatrix}$.

The give both the algebraic and geometric multiplicity of each of the eigenvalues of A.

Solution: The characteristic polynomial is $p(\lambda) = -\lambda(1-\lambda)^2$ so the eigenvalues are $\lambda = 0$ with algebraic multiplicity 1 and $\lambda = 1$ with algebraic multiplicity 2. The eigenvalue $\lambda = 0$ yields the eigenvector $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ giving geometric multiplicity 1, and the eigenvalue $\lambda = 1$ yields the eigenvector $\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$ giving geometric multiplicity 1.

(b) Compute the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$. The give both the algebraic and geometric multiplicity of each of the eigenvalues of A.

Solution: The characteristic polynomial is $p(\lambda) = (1 - \lambda)(2 - \lambda)(3 - \lambda)$ so the eigenvalues are 1,2, and 3 each with algebraic multiplicity 1 which implies that their geometric multiplicity is also 1. The eigenvector for $\lambda = 1$ is $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$, the eigenvector for $\lambda = 2$ is $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$, and the eigenvector for $\lambda = 3$ is $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

(c) Consider the two matrices

$$A = \begin{bmatrix} 3 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & -1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Compute the eigenvalues and eigenvectors of each and give both the algebraic and geometric multiplicity of each.

Solution: By using the block diagonal structure it follows that he characteristic polynomial for both matrices is $p(\lambda) = (\lambda - 2)^4$. So $\lambda = 2$ is the only eigenvalue for both matrices. For the first matrix we get two eigenvectors

$$\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix},$$

so the geometric multiplicity is 2. For the second matrix $\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ is the only eigenvector so the geometric multiplicity is 1.

(d) Let $A \in \mathbb{R}^{n \times n}$ and let $P \in \mathbb{R}^{n \times n}$ be invertible. Show that if $\lambda \in \mathbb{R}$ is an eigenvalue of A with associated eigenvector x, then λ is an eigenvalue of $B = PAP^{-1}$ associated with eigenvector $y = Px$.

Solution: $By = PAP^{-1}Px = PAx = \lambda Px = \lambda y$.

(e) Let $u \in \mathbb{R}^n$ be such that $u^T u = 1$ and set $A = I_n - uu^T$. Compute the eigenvalues of A and their associated eigenvectors. In particular, show that A is diagonalizable.

Solution: First observe that $A^2 = (I - uu^T)(I - uu^T) = I - uu^T - uu^T + uu^T uu^T = (I - uu^T) = A$. Since $Au = 0$, u is an eigenvector with eigenvalue 0. Observe that $0 = Aw = w - (u^T w)u$ which implies that w is a multiple of u. Let $\{u, w_2, w_3, \ldots, w_n\}$ be a basis for $\mathbb{R}^n$. Then $Aw_i \neq 0$, $i - 2, \ldots, n$ since if $Aw_i = 0$ then w_i is a multiple of u which contradicts the linear independence of the vectors $\{u, w_2, w_3, \ldots, w_n\}$. Set $\hat{w}_i = Aw_i = (I - uu^T)w_i$, $i = 2, \ldots, n$. Then $A\hat{w}_i = AAw_i = A^2 w_i = Aw_i = \hat{w}_i$ so that every $\hat{w}_i$ is an eigenvector of A with eigenvalue

1. Moreover, the vectors $\hat{w}_i,\ i = 2, \ldots, n$ are linearly independent since otherwise there exist μ_i such that $0 = \mu_2 \hat{w}_2 + \cdots + \mu_n \hat{w}_n = (I - uu^T)(\mu_2 w_2 + \cdots + \mu_n w_n)$ so that $\mu_2 w_2 + \cdots + \mu_n w_n$ is a multiple of u which again contradicts the linear independence of the vectors $\{u, w_2, w_3, \ldots, w_n\}$. Hence the geometric multiplicity of the eigenvalue 1 is $(n-1)$ and the geometric multiplicity of the eigenvalue 0 is 1. Therefore, A is diagonalizable with $A = PDP^{-1}$ where $P = [u, w_2, w_3, \ldots, w_n]$ and

$$D = \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ \vdots & \ddots & & \ldots & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & \ldots & \ldots & & 1 \end{bmatrix}.$$