Perspectives on Duality in Convex Optimization

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Joint work with
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McGill University
Discrete Mathematics and Optimization Seminar
February 20, 2017

Dual pairs of problems

A prototype problem: $A \in \mathbb{R}^{m \times n},\ b \in \mathbb{R}^m$

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The gauge dual problem:

$$\mathcal{D}_{\mathrm{G}} \qquad \begin{array}{ll} \inf & \left\|A^T z\right\|_\infty \\ \text{s.t.} & \langle b, z\rangle - \tau\|z\|_2 \ge 1. \end{array}$$

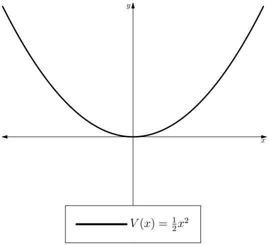
Piecewise Linear-Quadratic Penalties

$$\phi(x) := \sup_{u \in U} [\langle x, u \rangle - \frac{1}{2} u^T B u]$$

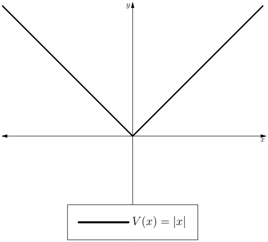
$U \subset \mathbb{R}^n$ is nonempty, closed and convex with $0 \in U$.
$B \in \mathbb{R}^{n \times n}$ is symmetric positive semi-definite.

Examples:
Norms, gauges, support functions, least-squares, Huber density

PLQ Densities: Gauss, Laplace, Huber, Vapnik



Gauss



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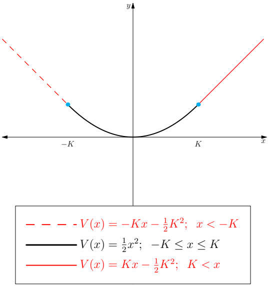
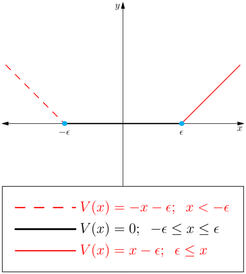


Huber



Vapnik

Convex Sets

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A subset C of $\mathbb{R}^n$ is convex if

$$[x, y] \subset C \quad \forall\, x, y \in C,$$

where

$$[x, y] := \{(1 - \lambda x) + \lambda y \,|\, 0 \leq \lambda \leq 1\}$$

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A subset K of $\mathbb{R}^n$ is convex if

$$\lambda K \subset K \;\; \forall\, \lambda > 0 \;\; \text{and } K + K \subset K.$$

Convex functions and the epigraphical perspective

A function $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ is said to be convex if

$$\operatorname{epi} f := \{(x, \mu) \mid f(x) \leq \mu\},$$

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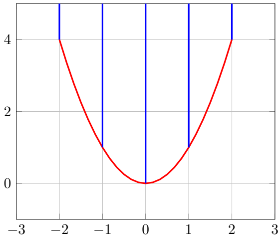

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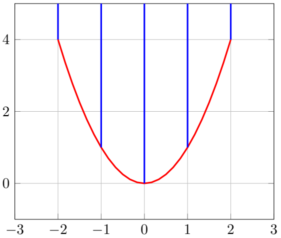

f is lower semi-continuous (lsc) $\iff$ $\operatorname{epi}(f)$ is closed

Linear transformations and their pre-images

If $A, B^T \in \mathbb{R}^{m \times n}$, then both

$$\begin{aligned} AC &:= \{Ax \mid x \in C\} \subset \mathbb{R}^m \text{ and} \\ B^{-1}C &:= \{y \mid By \in C\} \subset \mathbb{R}^m \end{aligned}$$

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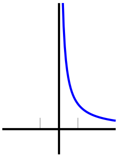

Figure : The graph of $1/x$

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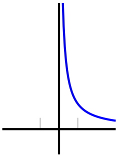

Figure : The graph of $1/x$

Convex hull: The convex hull of $S \subset \mathbb{R}^n$ is the intersection of all convex sets in $\mathbb{R}^n$ containing S, denoted $\operatorname{conv}(S)$.

Affine sets and relative interior

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Relative interior: The relative interior of a convex set is the interior relative to its affine hull:

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Properties: Let $C \subset \mathbb{R}^n$ be convex and $A, B^T \in \mathbb{R}^{m \times n}$, then

$$\begin{aligned} A \operatorname{ri}(C) &= \operatorname{ri}(AC) \quad \text{and} \\ B^{-1} \operatorname{ri}(C) &= \operatorname{ri}(B^{-1}C), \quad \text{whenever } B^{-1}\operatorname{ri}(C) \neq \emptyset. \end{aligned}$$

The Hahn-Banach Theorem

Hyperplanes: Affine sets of co-dimension 1, or equivalently, any set of the form

$$\{x \mid \langle z, x\rangle = \beta\}$$

for some $\beta \in \mathbb{R}$ and non-zero $z \in \mathcal{L}$.

The Hahn-Banach Theorem: Let M be a nonempty affine set such that

$$M \cap \operatorname{ri} C = \emptyset.$$

Then there is a hyperplane H such that

$$M \subset H \quad \text{and} \quad H \cap \operatorname{ri} C = \emptyset.$$

Supporting hyperplanes

If $\overline{x} \in \operatorname{rbdry}(C) := \operatorname{cl} C \setminus \operatorname{ri} C$, then there is a hyperplane H containing $\overline{x}$ that does not meet the relative interior of C, or equivalently,

$$\exists\, z \text{ s.t. } \langle z, x\rangle < \langle z, \overline{x}\rangle \quad \forall\, x \in \operatorname{ri} C. \tag{1}$$

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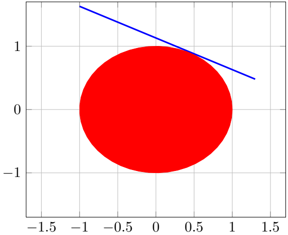

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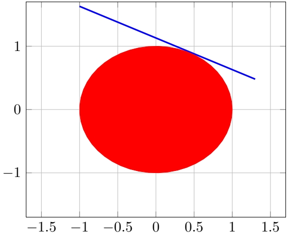

Question: Given $z \in \mathbb{R}^n$, does it define a supporting hyperplane to C and what are the associated support points.

Support functions

The support function for a set $S \subset \mathbb{R}^n$ is given by

$$\sigma_S(z) := \sup_{x \in S} \langle z, x \rangle.$$

It is straightforward to show that

$$\sigma_S(z) = \sigma_C(z), \quad \text{where} \quad C := \overline{\mathrm{conv}}(S).$$

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Hörmander's Theorem: $\sigma : \mathbb{R}^n \to \bar{\mathbb{R}}_+ := \mathbb{R}_+ \cup \{+\infty\}$ lsc.

σ is subadditive $\iff$ epi(σ) is a closed cvx cone $\iff$ $\sigma = \sigma_C$,

where $C := \{ z \mid \langle z, x \rangle \leq f(x) \; \forall \, x \} = \{ z \mid \langle z, x \rangle \leq 1 \; \forall \, f(x) \leq 1 \}$.

Convex functions and the epigraphical perspective

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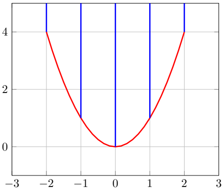

$\operatorname{epi}(f)$

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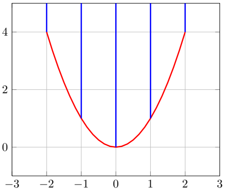

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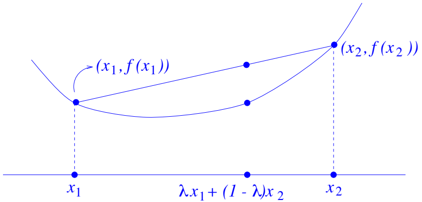

$$f((1-\lambda)x_1 + \lambda x_2) \leq (1-\lambda) f(x_1) + \lambda f(x_2)$$

$$\forall\ x_1, x_2 \in \text{dom}\, f \text{ and } \lambda \in [0, 1]$$

$$\text{dom}\, f := \{x \mid f(x) < \infty\}$$

Coordinate inf-projection of a convex set

Let $C \subset \mathbb{R}^{m+1}$ be a convex set such that the projection of C onto its last coordinate is bounded below. Define $f : \mathbb{R}^m \to \bar{\mathbb{R}}\}$ by

$$f(x) := \inf \{\overline{x}_{m+1} \mid \exists\, \overline{x} \in C \text{ s.t. } \overline{x} = (x, \overline{x}_{m+1})\},$$

where, again, the infimum over the empty set is $+\infty$.

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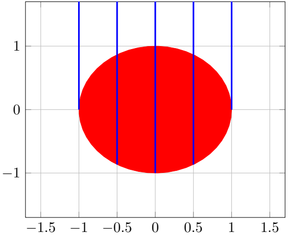
$$\mathrm{epi}\,(f) = C + (\{0\}^m \times \mathbb{R}_+)$$



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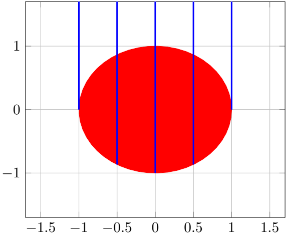

Example: $f(x) := \inf\limits_{(x,\mu)\in \text{epi}\,(f)} \mu$

Inf-projection: $h(x) := \inf_y f(y, x)$

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The support function for epi (f): the convex conjugate

$$\begin{aligned}
\sigma_{\operatorname{epi} f}((z,-1)) &= \sup_{f(x)\le\mu} \langle (z,-1),(x,\mu)\rangle \\
&= \sup_{f(x)\le\mu} [\langle z,x\rangle - \mu] \\
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$$\iff f(x) \geq f(\overline{x}) + \langle z, x-\overline{x}\rangle \quad \forall\, x \in \mathbb{R}^n$$

The support function for epi (f): the convex conjugate

$$\begin{aligned}
\sigma_{\mathrm{epi}\, f}\left((z,-1)\right) &= \sup_{f(x)\leq\mu} \langle (z,-1),(x,\mu)\rangle \\
&= \sup_{f(x)\leq\mu} \left[\langle z,x\rangle - \mu\right] \\
&= \sup_{x}\left[\langle z,x\rangle - f(x)\right] \\
&=: f^*(z)
\end{aligned}$$

Subgradients:

$$\overline{x} \in \operatorname*{argmax}_x\left[\langle z,x\rangle - f(x)\right] \iff (z,-1) \text{ supports epi}\,(f) \text{ at } (\overline{x}, f(\overline{x})).$$

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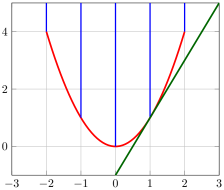
$$\iff f(x) \geq f(\overline{x}) + \langle z, x-\overline{x}\rangle \quad \forall\, x \in \mathbb{R}^n$$

$z \in \partial f(\overline{x})$, the subdifferential of f at $\overline{x}$.

The conjugate and subgradients



$$\partial f(\overline{x}) \text{ is a singleton} \iff \partial f(\overline{x}) = \{\nabla f(\overline{x})\}.$$

The conjugate and subgradients

$$f^*(z) \geq \langle z, x \rangle - f(x) \quad \forall \;\; x \in \operatorname{dom}(f) \text{ and } z \in \mathbb{R}^n$$

$$\iff \quad f(x) \geq \langle z, x \rangle - f^*(z) \qquad \forall \;\; z \in \operatorname{dom}(f^*) \text{ and } x \in \mathbb{R}^n$$

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so $\forall\, x \in \mathrm{dom}\,(\partial f) := \{x \,|\, \partial f(x) \neq \emptyset\}$ and $z \in \partial f(x)$,

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So $f(x) = f^{**}(x)$ on $\operatorname{dom}(\partial f)$, where $\operatorname{ri}\operatorname{dom}(f) \subset \operatorname{dom}(\partial f)$.
Consequently $f^{**} = \operatorname{cl} f$, so if $f = \operatorname{cl} f$, $\partial f^* = (\partial f)^{-1}$.

The convex indicator function

$C \subset \mathbb{R}^n$ non-empty closed convex

$$\delta_C\,(x) := \begin{cases} 0 & ,\ x \in C, \\ +\infty & ,\ x \notin C \end{cases}$$

$$\delta_C^*(z) = \sigma_C\,(z)$$

$$\begin{aligned} \partial\delta_C\,(x) &= \{z \mid \langle z, y - x\rangle \leq 0 \ \ \forall\, y \in C\} \quad (x \in C) \\ &=: N\,(x \,|\, C) \qquad \text{the normal cone to } C \text{ at } x \\ &= \text{set of supporting vectors to } C \text{ at } x \end{aligned}$$

The conjugate under inf-projection

Let $F : \mathbb{R}^n \times \mathbb{R}^m \to \bar{\mathbb{R}}$ and define the following optimal value function by inf-projection:

$$p(y) := \inf_x F(x, y).$$

Then

$$\begin{aligned} p^*(z) &= \sup_y [\langle z, y\rangle - p(y)] \\ &= \sup_y [\langle z, y\rangle - \inf_x F(x, y)] \\ &= \sup_y \sup_x [\langle z, y\rangle - F(x, y)] \\ &= \sup_{(x,y)} [\langle (0, z), (x, y)\rangle - F(x, y)] \\ &= F^*(0, z) \end{aligned}$$

The subdifferential under inf-projection

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$$\Longleftrightarrow$$

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Let $F\colon \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}$ and define the following optimal value functions by inf-projection:

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This set-up yields the primal-dual pair

$$p(0) = \inf_x \; F(x,0) \quad \text{and} \quad p^{**}(0) = \sup_z \; -F^*(0,z) \;\; = \;\; -q(0).$$

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$$p(0) \;\geq\; p^{**}(0) \;=\; -q(0) \text{ always holds}$$

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1. If $0 \in \operatorname{ri}(\operatorname{dom} p)$, then $p(0) = -q(0)$ and the infimum $q(0)$ is attained, if finite.

   Similarly, if $0 \in \operatorname{ri}(\operatorname{dom} q)$, then $p(0) = -q(0)$ and the infimum $p(0)$ is attained, if finite.

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4. Optimal solutions are characterized by

$$\left.\begin{array}{l} \bar{x} \in \mathrm{argmin}_x \;\; F(x,0) \\ \bar{y} \in \mathrm{argmax}_z \;\; -F^*(0,z) \\ F(\bar{x},0) = -F^*(0,\bar{z}) \end{array}\right\} \iff (0,\bar{z}) \in \partial F(\bar{x},0) \iff (\bar{x},0) \in \partial F^*(0,\bar{z}).$$

Lagrangian Duality

The Lagrangian function:

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&= \langle A^T z + w, x\rangle - [h^*(z) + g^*(w)]
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Example

Fenchel-Rockafellar Duality: $F(x, y) = h(Ax + y) + g(x)$

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$$\mathcal{P} \qquad \begin{array}{rl} \min & \|x\|_1 \\ \text{s.t.} & \|Ax-b\|_2 \leq \tau \end{array}$$

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$$\mathcal{D}_{\mathrm{L}} \qquad \begin{array}{l} \sup \ \langle b, z\rangle - \tau\|z\|_2 \\ \text{s.t. } \left\|A^T z\right\|_\infty \le 1. \end{array}$$

Gauge Duality

Dual Norms and Polars

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Properties:

1. $(S^\circ)^\circ = \overline{\mathrm{conv}}\,(S \cup \{0\})$
2. K is a close convex cone ($\lambda K \subset K \;\forall\, \lambda > 0,\; K + K \subset K$), then

$$K^\circ = \{z \mid \langle z, x\rangle \le 0 \;\forall\, x \in K\}.$$

Minkowski (gauge) functionals and polarity

$0 \in C \subset \mathbb{R}^n$ nonempty closed convex

$$\gamma_C(x) := \inf\{t \mid 0 \le t,\ x \in tC\},$$

where the infimum over the empty set is $+\infty$.

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Gauge functions are sublinear, and so by Hörmander,

$$\gamma_C(x) = \sigma_D(x),$$

where

$$D = \{z \mid \langle z, x\rangle \leq 1\ \forall\ x \in C\} = C^\circ.$$

Polar Gauges

$$\kappa^\circ(y) = \sup\{\, \langle y, x\rangle \mid \kappa(x) \le 1 \,\} = \sigma_{\mathbb{U}_\kappa}(y)\,,$$

where $\mathbb{U}_\kappa := \{x \mid \kappa(x) \le 1\}$.

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The generalized Hölder inequality

$$\langle x, y \rangle \le \kappa(x) \cdot \kappa^\circ(y) \quad \forall x \in \operatorname{dom} \kappa, \ \forall y \in \operatorname{dom} \kappa^\circ,$$

is known as the *polar-gauge inequality.*

Gauge Duality

κ and ρ are gauges.

$$\min_x \quad \kappa(x) \qquad \text{s.t.} \quad \rho(b - Ax) \leq \tau, \tag{G$_p$}$$

$$\max_y \quad \langle b, y \rangle - \tau\rho^\circ(y) \qquad \text{s.t.} \quad \kappa^\circ(A^T y) \leq 1, \tag{L$_d$}$$

$$\min_y \quad \kappa^\circ(A^T y) \qquad \text{s.t.} \quad \langle b, y \rangle - \tau\rho^\circ(y) \geq 1. \tag{G$_d$}$$

Example: In $\mathcal{P}$ we set $\kappa(x) = \|x\|_1$ and $\rho(y) = \|y\|_2$

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Feasibility

Primal, Dual Domains:

$$\mathcal{F}_p := \{\, x \mid \rho(b - Ax) \le \tau \,\} \qquad \text{and} \qquad \mathcal{F}_d := \{\, y \mid \langle b, y\rangle - \tau\rho^\circ(y) \ge 1 \,\}\,.$$

$$\textbf{Feasibilty}: \quad \begin{array}{ll} \text{Primal} & \mathcal{F}_p \cap (\operatorname{dom}\kappa) \\ \text{Dual} & A^T\,\mathcal{F}_d \cap (\operatorname{dom}\kappa^\circ) \end{array}$$

$$\textbf{Relative Strict Feasibilty}: \quad \begin{array}{ll} \text{Primal} & \operatorname{ri}\mathcal{F}_p \cap (\operatorname{ri}\operatorname{dom}\kappa) \\ \text{Dual} & A^T\operatorname{ri}\mathcal{F}_d \cap (\operatorname{ri}\operatorname{dom}\kappa^\circ) \end{array}$$

$$\textbf{Strict Feasibilty}: \quad \begin{array}{ll} \text{Primal} & \operatorname{int}(\mathcal{F})_p \cap (\operatorname{ri}\operatorname{dom}\kappa) \\ \text{Dual} & A^T\operatorname{int}(\mathcal{F})_d \cap (\operatorname{ri}\operatorname{dom}\kappa^\circ) \end{array}$$

Gauge Duality

$$v_p \;=\; \min_{\rho(b-AX)\leq\tau} \kappa(x) \qquad\qquad v_d = \min_{\langle b,y\rangle - \tau\rho^\circ(y)\geq 1} \kappa^\circ(A^T y)$$

Theorem: (2014)

1. (Weak duality)
   If x and y are P-D feasible, then
   $$1 \leq v_p v_d \leq \kappa(x)\cdot\kappa^\circ(A^T y).$$

2. (Strong duality)
   If the dual (resp. primal) is feasible and the primal (resp. dual) is relatively strictly feasible, then $\nu_p\nu_d = 1$ and the gauge dual (resp. primal) attains its optimal value.

Freund (1987), Friedlander-Macedo-Pong (2014)

Gauge Duality and Sensitivity

$$v_p(y) := \inf_{\mu>0,\,x} \{\mu \mid \rho\,(b - Ax + \mu y) \le \tau,\ \kappa(x) \le \mu\}$$

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Variational framework:

$$F(w,\lambda,y) := -\lambda + \delta_{(\operatorname{epi}\rho)\times\mathbb{U}_\kappa}\left(W\begin{pmatrix} w \\ \lambda \\ y \end{pmatrix}\right), \quad W := \begin{pmatrix} -A & b & I \\ 0 & \tau & 0 \\ I & 0 & 0 \end{pmatrix}$$

$$F^*(w,\lambda,y) = \delta_{\operatorname{epi}\rho^\circ}\begin{pmatrix} y \\ -\sigma^{-1}(1+\lambda-\langle b,y\rangle) \end{pmatrix} + \kappa^\circ(w + A^T y)$$

Gauge Duality and Sensitivity

$$p(y) := \inf_{w,\lambda} F(w, \lambda, y)$$

$$\begin{aligned} v_p(y) := \inf_x \;& \mu \\ \text{s.t.}\;& \rho(b - Ax + \mu y) \le \tau \\ & \kappa(x) \le \mu \end{aligned}$$

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2. If the dual is strictly feasible and the primal is feasible, then the set of optimal solutions for the primal is nonempty and bounded with solutions given by $x^* = w^*/\lambda^*$, where

$$(w^*, \lambda^*) \in \partial v_d(0,0) \text{ and } \lambda^* > 0.$$

Gauge Duality and Optimality Conditions

Suppose both the gauge primal and dual are strictly feasible. Then the pair (x^*, y^*) is primal-dual optimal if and only

$$\sigma = \rho(b - Ax^*) \qquad \text{(primal activity)}$$

$$1 = \langle b, y^* \rangle - \sigma \rho^\circ(y^*) \qquad \text{(dual activity)}$$

$$\langle x^*, A^T y^* \rangle = \kappa(x^*) \cdot \kappa^\circ(A^T y^*) \qquad \text{(objective alignment)}$$

$$\langle b - Ax^*, y^* \rangle = \rho(b - Ax^*) \cdot \rho^\circ(y^*). \qquad \text{(constraint alignment)}$$

Gauge primal-dual recovery

Suppose that the gauge primal and dual are strictly feasible. If y is optimal for G_d, then for any $x \in \mathbb{R}^n$ the following conditions are equivalent:

(a) x is optimal for G_p;

(b) $\langle x, A^T y\rangle = \kappa(x) \cdot \kappa^\circ(A^T y)$ and $b - Ax \in \sigma\partial\rho^\circ(y)$;

(c) $A^T y \in \kappa^\circ(A^T y) \cdot \partial\kappa(x)$ and $b - Ax \in \sigma\partial\rho^\circ(y)$.

Gauge primal-dual recovery from the Lagrange dual

Suppose that the gauge dual G_d is strictly feasible and the primal G_p is feasible.

Let $\mathcal{L}G_d$ denote the Lagrange dual of G_d, and let ν_L denote its optimal value.

Then

$$z^* \text{ is optimal for } \mathcal{L}G_d \iff z^*/\nu_L \text{ is optimal for } G_p.$$

Perspective Duality

The Perspective-Polar Transform

$$
\begin{aligned}
f^\sharp(x,\xi) &:= (f^\pi)^\circ(x,\xi) \\
&= \sigma_{\mathrm{epi}\,(f^*)^\circ}\,(x,-\xi) \\
&= \gamma_{\mathrm{epi}\,(f^*)}\,(x,-\xi) \\
&= \inf\{\mu > 0 \mid \xi + \langle z, x\rangle \le \mu f(z),\ \forall z\}
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$f^\sharp$ is a gauge.

If f is a gauge, then $f^\sharp(x,\xi) = f^\circ(x) + \delta_{\mathbb{R}_-}(\xi)$.

The Perspective-Polar of a PLQ

Piecewise linear-quadratic (PLQ) functons:

$$g(y) := \sup_{u\in U} \left\{ \langle u, y\rangle - \tfrac{1}{2}\|Lu\|_2^2 \right\}, \quad U := \left\{ u \in \mathbb{R}^l \mid Wu \le w \right\},$$

$$\begin{aligned} g^{\sharp}(y,\mu) &= \delta_{\mathbb{R}_-}(\mu) + \max\left\{\gamma_U(y),\, -(1/2\mu)\|Ly\|^2\right\} \\ &= \delta_{\mathbb{R}_-}(\mu) + \max\left\{-(1/2\mu)\|Ly\|^2,\ \max_{i=1,\dots,k}\left\{ {W_i}^T y/w_i \right\}\right\}, \end{aligned}$$

where $W_1^T, \dots, W_k^T$ are the rows of W.

Perspective duality

Suppose $f : \mathbb{R}^n \to_+$ and $g : \mathbb{R}^m \to_+$ are closed, convex and nonnegative over their domains.

$$N_p \quad \min_{x} \quad f(x) \qquad \text{s.t.} \quad g(b - Ax) \leq \sigma,$$

$$N_d \quad \min_{y,\ \alpha,\ \mu} \quad f^\sharp(A^T y, \alpha) \qquad \text{s.t.} \quad \langle b, y \rangle - \sigma \cdot g^\sharp(y, \mu) \geq 1 - (\alpha + \mu)$$

The Perspective Duality for PLQ

Assume f is a gauge and g is PLQ, then

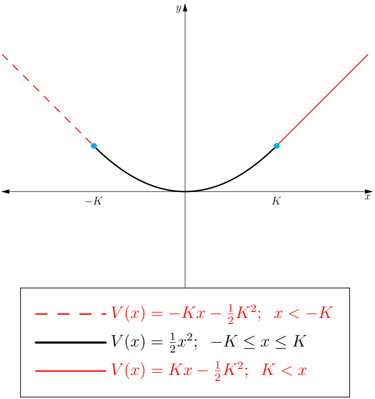
$$\begin{array}{ll} \min_{(y,\mu,\xi)} & f^\circ\big(A^T y\big) \\ \text{s.t.} & \langle b, y\rangle + \mu - \sigma\xi = 1 \\ & \quad Wy \le \xi w,\ \left\|\begin{bmatrix} 2Ly \\ \xi + 2\mu \end{bmatrix}\right\|_2 \le \xi - 2\mu \end{array}$$

Perspective Duality Numerics

$$\min_x \quad \|x\|_1$$

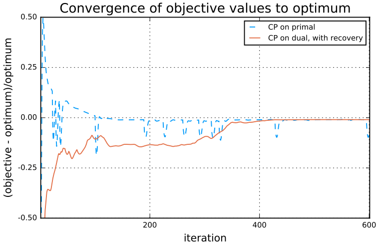
$$\text{s.t.} \quad \sum_{i=1}^{m} V((Ax-b)_i) \le \sigma,$$

where V is the Huber function

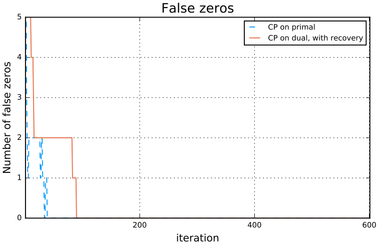


Experiment:
$m = 120$, $n = 512$, $\sigma = 0.2$, $\eta = 1$, and A is a Gaussian matrix. The true solution $x_{\text{true}} \in \{-1, 0, 1\}$ is a spike train which has been constructed to have 20 nonzero entries, and the true noise $b - Ax_{\text{true}}$ has been constructed to have 5 outliers.

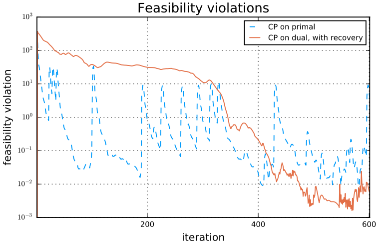
Perspective Duality Numerics



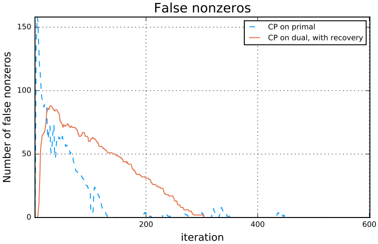
(a)



(c)



(b)



(d)

Chambolle- Pock (CP) algorithm

The Perspective Transform

$$f^\pi(x,\mu) := \operatorname{cl} \inf \ \tau \quad \text{s.t. } (x,\tau,\mu) \in \mathbb{R}_+[\operatorname{epi}(f) \times \{1\}]$$

$$= \begin{cases} \mu f(\mu^{-1}x), & \mu > 0 \\ f^\infty(x), & \mu = 0 \\ +\infty, & \mu < 0 \end{cases}$$

where

$$f^\infty(x) := \sup_{z \in \operatorname{dom}(f)} [f(x+z) - f(x)]$$

is the horizon function of f.

The Subdifferential of the Perspective

$$(f^\pi)^*(y,\xi) = \delta_{\mathrm{epi}\, f^*}\left((y,-\xi)\right)$$

$$\partial f^\pi(x,\mu) = \begin{cases} \{(z,-f^*(z)) \mid z \in \partial f(x/\mu)\} & \text{if } \mu > 0 \\ \\ \{(z,-\gamma) \mid (z,\gamma) \in \mathrm{epi}\, f^*,\;\; z \in \partial f^\infty(x)\} & \text{if } \mu = 0. \end{cases}$$

Properties of the Perspective

$$\sigma_{\mathrm{epi}\,h}\left((y,\mu)\right) = (h^*)^\pi(y,-\mu)$$

$$\sigma_{\mathrm{lev}_h(\tau)}(y) = \mathrm{cl}\inf_{\mu\geq 0}\left[\tau\mu + (h^*)^\pi(y,\mu)\right]$$