A Computational Approach to Schubert Varieties

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SAGE Days, 2007

Thanks!

Your contributions to SAGE development are an important part of this massive effort to unify free math software. Keep up the good work!

Goals for the talk

1. Connect with SAGE experts by introducing my research area: Schubert varieties.
    (a) The Flag Manifold
    (b) Schubert Cells and Schubert Varieties
    (c) Five Fun Facts about Schubert Varieties
2. Explain my dream for SAGE-TEX.
3. Describe some succesful computer proof techniques and some science fiction research.

Enumerative Geometry

Approximately 150 years ago... Grassmann, Schubert, Pieri, Giambelli, Severi, and others began the study of *enumerative geometry*.

Early questions:

- What is the dimension of the intersection between two general lines in $\mathbb{R}^2$?
- How many lines intersect two given lines and a given point in $\mathbb{R}^3$?
- How many lines intersect four given lines in $\mathbb{R}^3$?

Modern questions:

- How many points are in the intersection of 2,3,4,... Schubert varieties in general position?

Why Study Schubert Varieties?

1. It can be useful to see points, lines, planes etc as families of Schubert varieties with certain properties.

2. Schubert varieties provide interesting examples for test cases and future research in algebraic geometry, combinatorics, representation theory, symplectic geometry, and theoretical physics.

3. Applications in discrete geometry, computer graphics, and computer vision.

Vector Spaces

- $\boldsymbol{V}$ is a *vector space* over a field $\mathbb{F}$ if it is closed under addition and multiplication by scalars in $\mathbb{F}$.

- $\boldsymbol{B} = \{\boldsymbol{b_1}, \ldots, \boldsymbol{b_k}\}$ is a *basis* for $\boldsymbol{V}$ if for every $\boldsymbol{a} \in \boldsymbol{V}$ there exist unique scalars $\boldsymbol{c_1}, \ldots, \boldsymbol{c_k} \in \mathbb{F}$ such that

$$\boldsymbol{a} = \boldsymbol{c_1 b_1} + \boldsymbol{c_2 b_2} + \cdots + \boldsymbol{c_k b_k} = (\boldsymbol{c_1}, \boldsymbol{c_2}, \ldots, \boldsymbol{c_k}) \in \mathbb{F}^{\boldsymbol{k}}.$$

- The *dimension* of V equals the size of a basis.

- A *subspace* $\boldsymbol{U}$ of a vector space $\boldsymbol{V}$ is any subset of the vectors in $\boldsymbol{V}$ that is closed under addition and scalar multiplication.

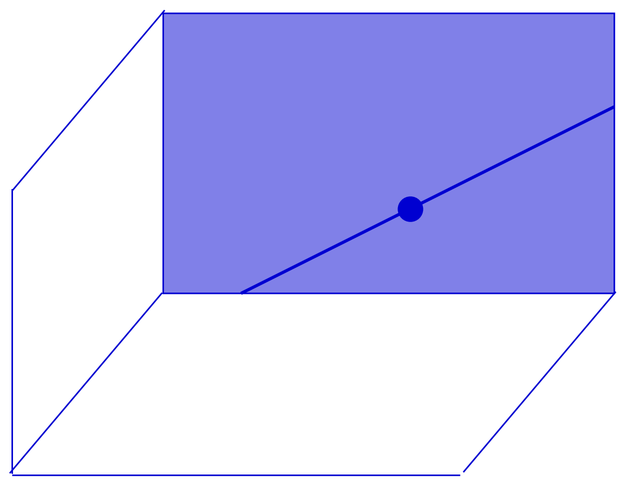
Fact. Any basis for $\boldsymbol{U}$ can be extended to a basis for $\boldsymbol{V}$.

The Flag Manifold

Defn. A *complete flag* $F_\bullet = (F_1, \ldots, F_n)$ in $\mathbb{C}^n$ is a nested sequence of vector spaces such that $\dim(F_i) = i$ for $1 \leq i \leq n$. $F_\bullet$ is determined by an ordered basis $\langle f_1, f_2, \ldots f_n \rangle$ where $F_i = \mathrm{span}\langle f_1, \ldots, f_i \rangle$.

Example.

$$F_\bullet = \langle 6e_1 + 3e_2, \quad 4e_1 + 2e_3, \quad 9e_1 + e_3 + e_4, \quad e_2 \rangle$$



The Flag Manifold

Canonical Form. Every flag can be represented as a matrix in row echelon form.

$$F_\bullet = \langle 6e_1 + 3e_2,\quad 4e_1 + 2e_3,\quad 9e_1 + e_3 + e_4,\quad e_2 \rangle$$

$$\approx \begin{bmatrix} 6 & 3 & 0 & 0 \\ 4 & 0 & 2 & 0 \\ 9 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 7 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\approx \langle 2e_1 + e_2,\quad 2e_1 + e_3,\quad 7e_1 + e_4,\quad e_1 \rangle$$

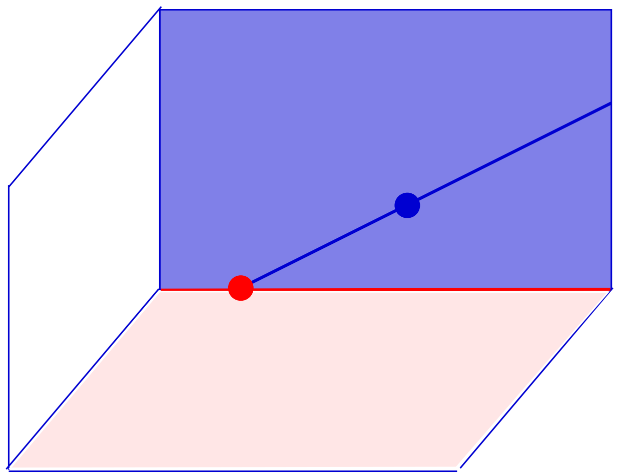
$\mathcal{F}l_n(\mathbb{C}) :=$ *flag manifold* over $\mathbb{C}^n = \{$complete flags $F_\bullet\}$

$= B \setminus GL_n(\mathbb{C}),\quad B =$ lower triangular mats.

Flags and Permutations

Example. $F_\bullet = \langle 2e_1+e_2,\ \ 2e_1+e_3,\ \ 7e_1+e_4,\ \ e_1\rangle \approx \begin{bmatrix} 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 7 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

Note. If a flag is written in canonical form, the positions of the leading 1's form a permutation matrix. There are 0's to the right and below each leading 1. This permutation determines the *position* of the flag $F_\bullet$ with respect to the reference flag $E_\bullet = \langle e_1,\ e_2,\ e_3,\ e_4\ \rangle$.



Many ways to represent a permutation

$$\begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} \\ \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{1} \end{bmatrix} = \mathbf{2341} = \begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{2} \\ \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} \end{bmatrix}$$

matrix notation

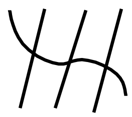
two-line notation

one-line notation

rank table

$$\begin{array}{cccc} * & \cdot & \cdot & \cdot \\ * & \cdot & \cdot & \cdot \\ * & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{array} =$$

1234



2341

$$= \mathbf{(1, 2, 3)}$$

diagram of a permutation

string diagram

reduced word

The Schubert Cell $C_w(E_\bullet)$ in $\mathcal{F}l_n(\mathbb{C})$

Defn. $C_w(E_\bullet)$ = All flags $F_\bullet$ with $\mathbf{position}(E_\bullet, F_\bullet) = w$

$$= \{F_\bullet \in \mathcal{F}l_n \mid \dim(E_i \cap F_j) = \mathrm{rk}(w[i,j])\}$$

Example. $F_\bullet = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 7 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \in C_{2341} = \left\{ \begin{bmatrix} * & 1 & 0 & 0 \\ * & 0 & 1 & 0 \\ * & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} : * \in \mathbb{C} \right\}$

Easy Observations.

- $\mathtt{dim}_{\mathbb{C}}(C_w) = l(w) = \#$ inversions of w.
- $C_w = w \cdot B$ is a B-orbit using the right B action, e.g.

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} b_{1,1} & 0 & 0 & 0 \\ b_{2,1} & b_{2,2} & 0 & 0 \\ b_{3,1} & b_{3,2} & b_{3,3} & 0 \\ b_{4,1} & b_{4,2} & b_{4,3} & b_{4,4} \end{bmatrix} = \begin{bmatrix} b_{2,1} & b_{2,2} & 0 & 0 \\ b_{3,1} & b_{3,2} & b_{3,3} & 0 \\ b_{4,1} & b_{4,2} & b_{4,3} & b_{4,4} \\ b_{1,1} & 0 & 0 & 0 \end{bmatrix}$$

The Schubert Variety $X_w(E_\bullet)$ in $\mathcal{F}l_n(\mathbb{C})$

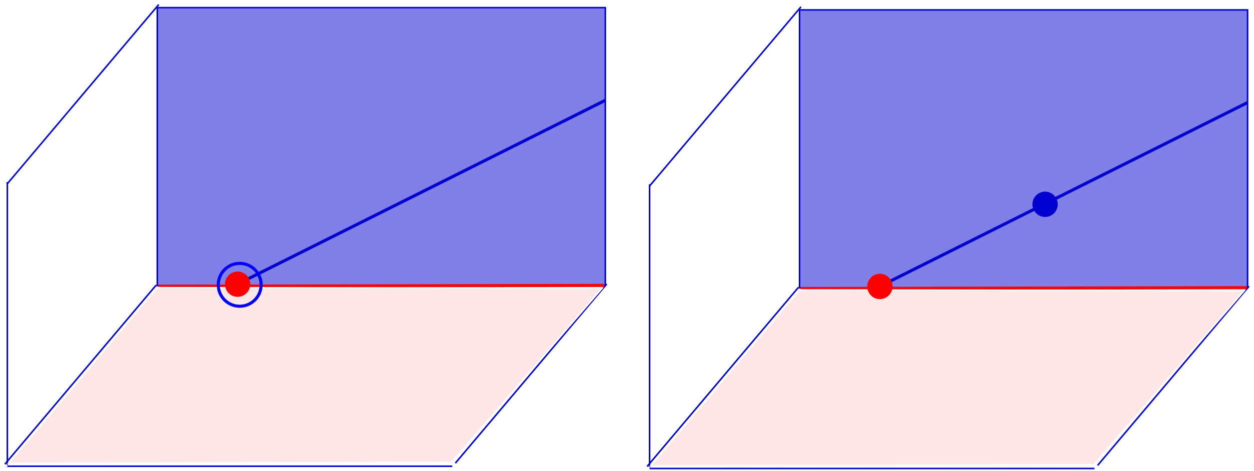
Defn. $X_w(E_\bullet)$ = Closure of $C_w(E_\bullet)$ under the Zariski topology

$$= \{F_\bullet \in \mathcal{F}l_n \mid \dim(E_i \cap F_j) \geq \mathrm{rk}(w[i,j])\}$$

where $E_\bullet = \langle e_1,\ e_2,\ e_3,\ e_4\ \rangle$.

Example. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & * & 1 & 0 \\ 0 & * & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \in X_{2341}(E_\bullet) = \overline{\left\{ \begin{bmatrix} * & 1 & 0 & 0 \\ * & 0 & 1 & 0 \\ * & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \right\}}$

Why?.



Five Fun Facts

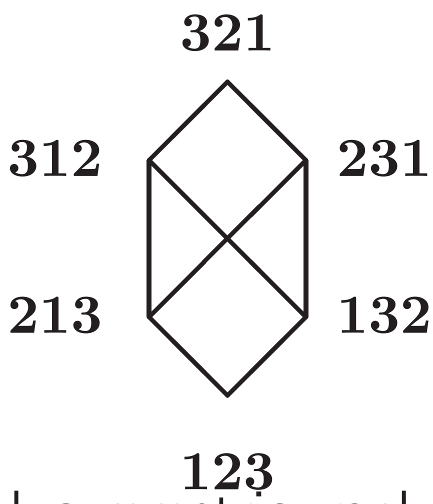
Fact 1. The closure relation on Schubert varieties defines a nice partial order.

$$X_w = \bigcup_{v \leq w} C_v \quad = \bigcup_{v \leq w} X_v$$

Bruhat order (Ehresmann 1934, Chevalley 1958) is the transitive closure of

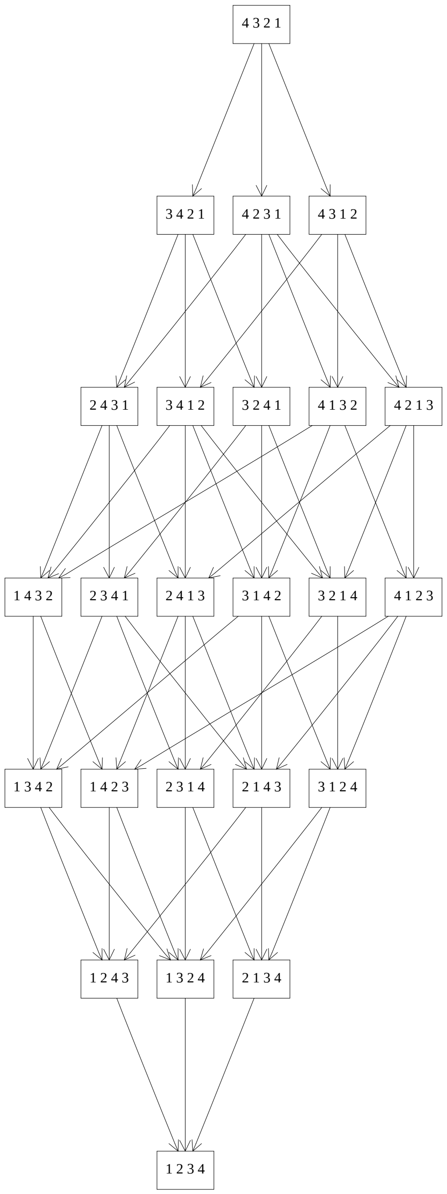
$$w < wt_{ij} \iff w(i) < w(j).$$

Example. Bruhat order on permutations in S_3.



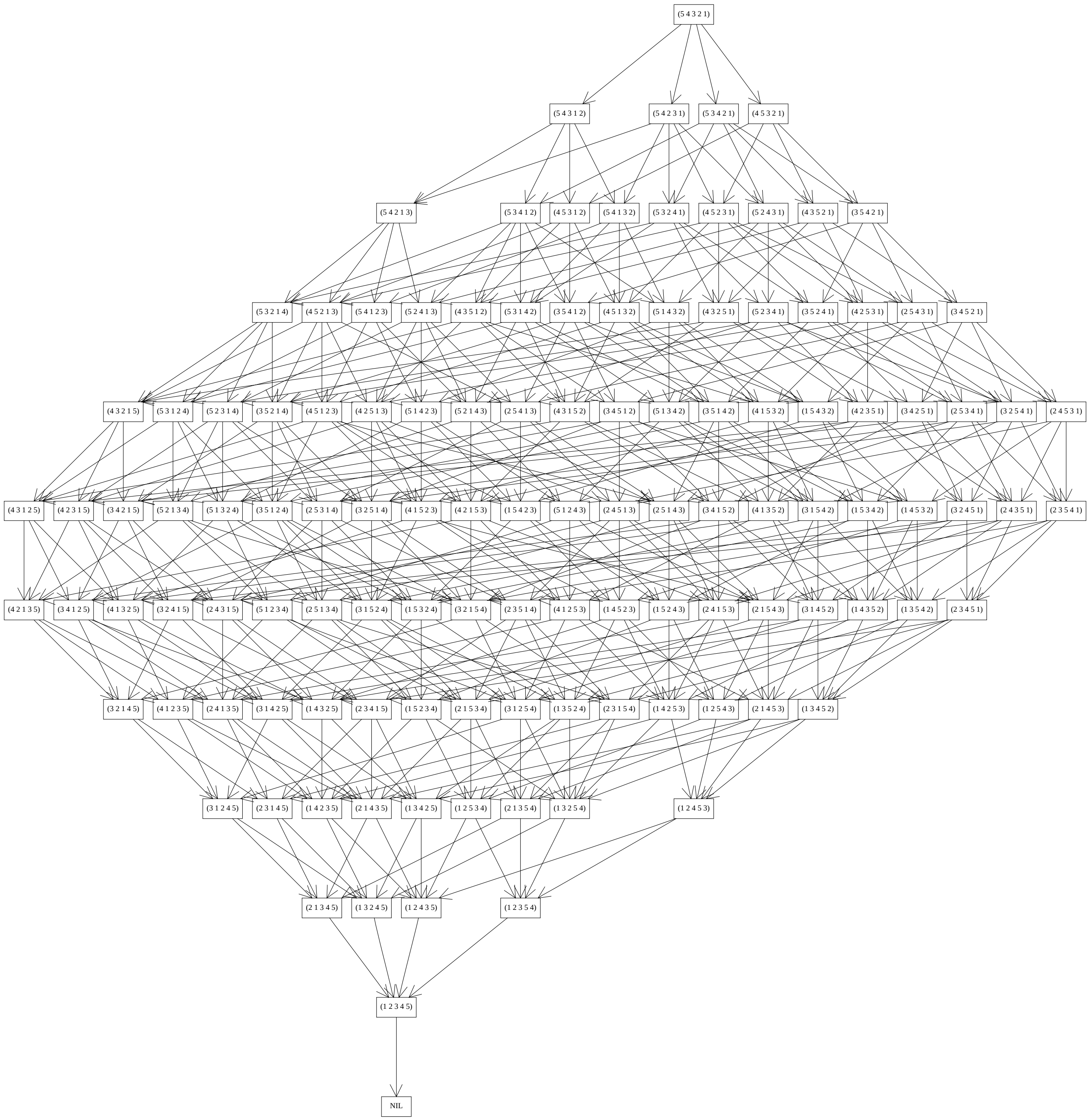
Observations. Self dual, rank symmetric, rank unimodal.

Bruhat order on S_4



Observations. Self dual, rank symmetric, rank unimodal.

Bruhat order on S_5



10 Fantastic Facts on Bruhat Order

1. Bruhat Order Characterizes Inclusions of Schubert Varieties
2. Contains Young's Lattice in S_∞
3. Nicest Possible Möbius Function
4. Beautiful Rank Generating Functions
5. $[x, y]$ Determines the Composition Series for Verma Modules
6. Symmetric Interval $[\hat{0}, w] \iff X(w)$ rationally smooth
7. Order Complex of (u, v) is shellable
8. Rank Symmetric, Rank Unimodal and k-Sperner
9. Efficient Methods for Comparison
10. Amenable to Pattern Avoidance

Five Fun Facts

Fact 2. There exists a simple criterion for characterizing singular Schubert varieties using pattern avoidance.

Theorem: Lakshmibai-Sandhya 1990 (see also Haiman, Ryan, Wolper)
X_w is non-singular $\iff$ w has no subsequence with the same relative order as 3412 and 4231.

	$w = 625431$	contains	$6241 \sim 4231$	$\implies X_{625431}$ is singular
Example:	$w = 612543$	avoids	4231 $\&3412$	$\implies X_{612543}$ is non-singular

Five Fun Facts

Consequences of Fact 2.

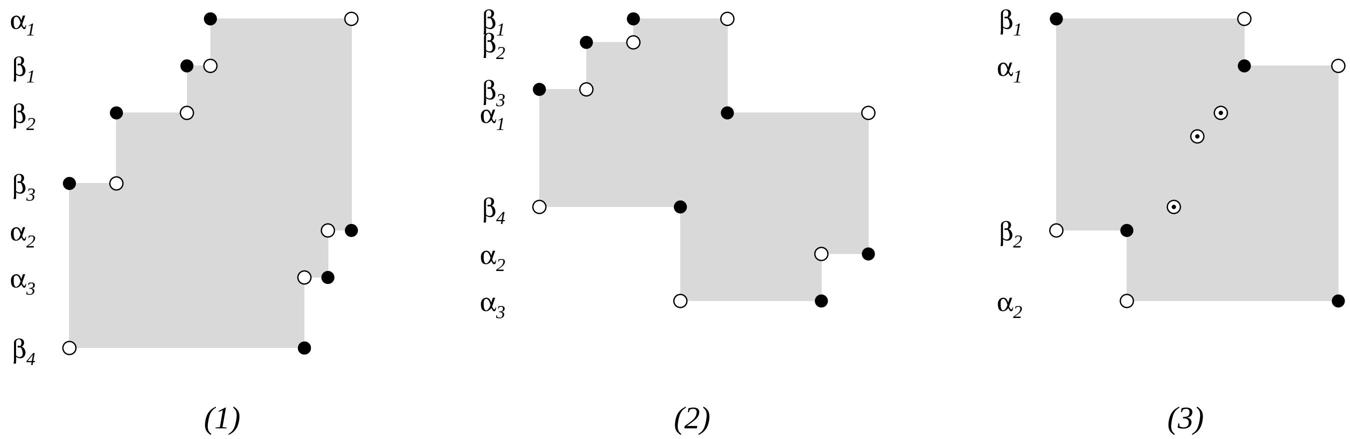
(Bona 1998, Haiman) Let v_n be the number of $w \in S_n$ for which $X(w)$ is non-singular. Then the generating function $V(t) = \sum_n v_n t^n$ is given by

$$\begin{aligned} V(t) =& \frac{1 - 5t + 3t^2 + t^2\sqrt{1-4t}}{1 - 6t + 8t^2 - 4t^3} \\ =& t + 2t^2 + 6t^3 + 22t^4 + 88t^5 + 366t^6 + 1552t^7 + 6652t^8 + O(t^9). \end{aligned}$$

(Billey-Postnikov 2001) Generalized pattern avoidance to all semisimple simply-connected Lie groups G and characterized smooth Schubert varieties X_w by avoiding these generalized patterns. Only requires checking patterns of types $A_3, B_2, B_3, C_2, C_3, D_4, G_2$.

Five Fun Facts

Fact 3. (Billey-Warrington, Kassel-Lascoux-Reutenauer, Manivel 2000)
We have $x \in$max-sing $(w) \iff x = w \cdot (\alpha_1, \ldots, \alpha_m, \beta_k, \ldots, \beta_1)$ corresponding to a 4231 or 3412 or 45312 pattern of the following form



Here ○'s denote 1's in w, ●'s denote 1's in x.

Open Problem. Describe the maximal singular locus of a Schubert variety for other semisimle Lie groups using generalized pattern avoidance.

Five Fun Facts

Fact 4. There exists a simple criterion for characterizing Gorenstein Schubert varieties using modified pattern avoidance.

Theorem: Woo-Yong (Sept. 2004)

X_w is Gorenstein $\iff$

- w avoids 31542 and 24153 *with Bruhat restrictions* $\{t_{15}, t_{23}\}$ and $\{t_{15}, t_{34}\}$

- for each descent d in w, the associated partition $\lambda_d(w)$ has all of its inner corners on the same antidiagonal.

Five Fun Facts

Fact 5. Schubert varieties are useful for studying the cohomology ring of the flag manifold.

Theorem (Borel): $H^*(\mathcal{F}l_n) \cong \dfrac{\mathbb{Z}[x_1, \ldots, x_n]}{\langle e_1, \ldots e_n \rangle}$.

- The symmetric function $e_i = \displaystyle\sum_{1 \le k_1 < \cdots < k_i \le n} x_{k_1} x_{k_2} \ldots x_{k_i}$.
- $\{[X_w] \mid w \in S_n\}$ form a basis for $H^*(\mathcal{F}l_n)$ over $\mathbb{Z}$.

Question. What is the product of two basis elements?

$$[X_u] \cdot [X_v] = \sum [X_w] c_{uv}^w.$$

Cup Product in $H^*(\mathcal{F}l_n)$

Answer. Use Schubert polynomials! Due to Lascoux-Schützenberger, Bernstein-Gelfand-Gelfand, Demazure.

- BGG: If $\mathfrak{S}_w \equiv [X_w] \mathrm{mod} \langle e_1, \ldots e_n \rangle$ then

$$\frac{\mathfrak{S}_w - s_i \mathfrak{S}_w}{x_i - x_{i+1}} \equiv [X_{ws_i}] \text{ if } l(w) < l(ws_i)$$

$$[X_{id}] \equiv x_1^{n-1} x_2^{n-2} \cdots x_{n-1} \equiv \prod_{i>j} (x_i - x_j) \equiv \ldots$$

  Here $\mathbf{deg}[X_w] = \mathbf{codim}(X_w)$.

- LS: Choosing $[X_{id}] \equiv x_1^{n-1} x_2^{n-2} \cdots x_{n-1}$ works best because product expansion can be done without regard to the ideal!

Schubert polynomials for S_4

$$\mathfrak{S}_{w_0(1234)} = 1$$
$$\mathfrak{S}_{w_0(2134)} = x_1$$
$$\mathfrak{S}_{w_0(1324)} = x_2 + x_1$$
$$\mathfrak{S}_{w_0(3124)} = x_1^2$$
$$\mathfrak{S}_{w_0(2314)} = x_1 x_2$$
$$\mathfrak{S}_{w_0(3214)} = x_1^2 x_2$$
$$\mathfrak{S}_{w_0(1243)} = x_3 + x_2 + x_1$$
$$\mathfrak{S}_{w_0(2143)} = x_1 x_3 + x_1 x_2 + x_1^2$$
$$\mathfrak{S}_{w_0(1423)} = x_2^2 + x_1 x_2 + x_1^2$$
$$\mathfrak{S}_{w_0(4123)} = x_1^3$$
$$\mathfrak{S}_{w_0(2413)} = x_1 x_2^2 + x_1^2 x_2$$
$$\mathfrak{S}_{w_0(4213)} = x_1^3 x_2$$
$$\mathfrak{S}_{w_0(1342)} = x_2 x_3 + x_1 x_3 + x_1 x_2$$
$$\mathfrak{S}_{w_0(3142)} = x_1^2 x_3 + x_1^2 x_2$$
$$\mathfrak{S}_{w_0(1432)} = x_2^2 x_3 + x_1 x_2 x_3 + x_1^2 x_3 + x_1 x_2^2 + x_1^2 x_2$$
$$\mathfrak{S}_{w_0(4132)} = x_1^3 x_3 + x_1^3 x_2$$
$$\mathfrak{S}_{w_0(3412)} = x_1^2 x_2^2$$
$$\mathfrak{S}_{w_0(4312)} = x_1^3 x_2^2$$
$$\mathfrak{S}_{w_0(2341)} = x_1 x_2 x_3$$
$$\mathfrak{S}_{w_0(3241)} = x_1^2 x_2 x_3$$
$$\mathfrak{S}_{w_0(2431)} = x_1 x_2^2 x_3 + x_1^2 x_2 x_3$$
$$\mathfrak{S}_{w_0(4231)} = x_1^3 x_2 x_3$$
$$\mathfrak{S}_{w_0(3421)} = x_1^2 x_2^2 x_3$$
$$\mathfrak{S}_{w_0(4321)} = x_1^3 x_2^2 x_3$$

Cup Product in $H^*(\mathcal{F}l_n)$

Key Feature. Schubert polynomials have distinct leading terms, therefore expanding any polynomial in the basis of Schubert polynomials can be done by linear algebra just like Schur functions.

Buch: Fastest approach to multiplying Schubert polynomials uses Lascoux and Schützenberger's transition equations. Works up to about $n = 15$.

Draw Back. Schubert polynomials don't prove c_{uv}^{w}'s are nonnegative (except in special cases).

Some Recommended Further Reading on Schubert varieties

1. "Schubert Calculus" by S. L. Kleiman; Dan Laksov The American Mathematical Monthly, Vol. 79, No. 10. (Dec., 1972), pp. 1061-1082.

2. "Young Tableaux" by William Fulton, London Math. Soc. Stud. Texts, Vol. 35, Cambridge Univ. Press, Cambridge, UK, 1997.

3. "The Symmetric Group" by Bruce Sagan, Wadsworth, Inc., 1991.

4. "Numerical Schubert calculus" by Birkett Huber, Frank Sottile, and Bernd Sturmfels, Journal of Symbolic Computation, 26, (1998) pp. 767-788.

5. "Determining the Lines Through Four Lines" by Michael Hohmeyer and Seth Teller, Journal of Graphics Tools, 4(3):11-22, 1999.

6. "Flag arrangements and triangulations of products of simplices" by Sara Billey and Federico Ardila, to appear in Adv. in Math.

7. "A Littlewood-Richardson rule for two-step flag varieties Revised version " by Izzet Coskun, preprint.

Dreams for Schubert Computations

Most of the tools we need are already in SAGE, like linear algebra packages, tools for solving polynomial equations, graphviz, etc.
Here is what we need to further research in this area:

1. *Poset functions*: fast comparison algorithms, intervals, rank generating functions, Möbius functions, test for lattice structure, test for poset isomorphism, (see Stanley "Enumerative Combinatorics" vol. 1)

2. *Symmetric group tools*: multiplication, cycle notation, representation theory tools, symmetric functions, Schubert polynomials (see Sagan "The Symmetric Group" and/or Fulton "Young Tableaux . . . ")

3. *Coxeter/Weyl group tools*: Build elements from a Coxeter graph, reduced expressions, length, descents, assents, Kazhdan-Lusztig polynomials, representation theory tools, Schubert polynomials, representation theory tools. (see "Combinatorics of Coxeter Groups" by Bjorner and Brenti)

4. *Pattern Avoidance*: Zeilberger's tools for efficient enumeration of pattern avoiding permutations, Billey's tools for pattern avoidance learning alorithms, patterns characterizing smoothness, rational smoothness, vexillary, freely braided, fully commutative, Boolean elements, etc. See also work of Vince Vatter.

Dreams for SAGE-TEX

SAGE + LaTeX = a better way to write about mathematics

Questions.

1. Could SAGE take in a whole latex file, macros and all, and convert it to readable html?

2. Could SAGE recognize embedded code in LaTeX? E.g.

```
\sage{
L = [[cos(pi*i/100),sin(pi*i/100)] for i in range(200)]
p = polygon(L, rgbcolor=(1,1,0))
p.save()           ## or   p.show()
}
```

3. Could SAGE allow a reader to easily modify an example or do experiments and then export a new pdf file?

Dreams for SAGE-TEX

My Goals for SAGE-TEX.

1. Write a monograph entitled "Computational Aspects of Schubert Varieties" with an emphasis on experimentation and calculations.

2. Combine the notion of *Wikibooks* with SAGE-TEXso others can easily improve on what I write, add new material, make corrections.

3. Use SAGE-TEXas a presentation tool for making slides.

Computer proofs

A computer-assisted proof is a mathematical proof that has been at least partially generated by computer. (From wikipedia.org)

1. *Four Color Theorem*: Any map can be colored by at most 4 colors so no two contries with a common boarder have the same color.

   (a) K. Appel and W. Haken, Every planar map is four colorable, Contemporary Math. 98 (1989). Announced in 1977.

   (b) N. Robertson, D. P. Sanders, P. D. Seymour and R. Thomas, The four colour theorem, J. Combin. Theory Ser. B. 70 (1997), 2-44.

2. *Kepler's Conjecture*: What is the densest arrangement of spheres in space? Answer: $\frac{\pi}{\sqrt{18}}$ Announced in 1998 by Thomas Hales. See articles by Hales and Ferguson in Discrete and Computational Geometry, 36 (2006), 1–265 for complete proof.

3. *Hypergeometric Series*: When does a hypergeometric series have a closed form? See "A = B" by by Petkovsek, Wilf and Zeilberger (1996) on-line.

4. *Smooth Schubert Varieties Theorem*: See Billey and Postnikov "Smoothness of Schubert Varieties via Patterns in Root Systems", Advances in Applied Math v. 34 (2005) p. 447-466.

Computer proofs

Techniques.

1. Exhaustive search.
2. Formal symbolic manipulation.
3. Universal comaparison via Sloane's Online Encylopedia.
4. Proof by example.
5. Estimation.

Open Problem. How can we recognize a theorem which is amenable to computer proof?

Computer Guessing

Question. How can we use a computer to guess the general form of a family of polynomials or rational functions? Say $f(j,k,n)$ is a (non-polynomial) function whose output is a linear polynomial in n variables. Can we guess the general form so a human can prove the formula by induction?

Example. $j = 2$, $k = 4$, $n = 7$,

$$x_2 + x_3 + 2x_4 + 2x_5 + 2x_6 + 2x_7$$

Guess. $\sum_{i=j}^{n} x_i + \sum_{i=k}^{n} x_i$

Summary of Problems

1. What is the maximal Singular locus of a Schubert variety for an arbitrary semisimple Lie group?

2. How can we most efficiently detect interval pattern avoidance?

3. Could SAGE take in a whole latex file, macros and all, and convert it to readable html?

4. Could SAGE recognize embedded code in $\LaTeX$?

5. Could SAGE allow a reader to easily modify an example or do experiments and then export a new pdf file?

6. How can we recognize a theorem which is amenable to computer proof?

7. Can you find other techniques for computer verification, automated comparison of theorems, and "computer guessing"?