Affine colored partitions

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Outline

1. Examples of affine (colored) partitions
2. Weyl groups, Affine Weyl groups, and their quotients
3. Bijection from affine partitions to these quotients
4. Application: Characterizing palindromic elements and smooth elements in affine Grassmannians.
5. Open problems

Example: Affine partitions of type G_2

Let $\mathcal{P}(G_2)$ be the family of partitions $(\lambda_1 \leq \cdots \leq \lambda_k)$ with parts $\lambda_i \in \{1,2,3,4,5,6\}$ such that each consecutive pair $(\lambda_i, \lambda_{i+1})$ is an *allowed pair*:

$$\{(1,4),(1,5),(1,6),(2,5),(2,6),(3,5),(3,6),\\ (4,5),(4,6),(5,5),(5,6),(6,6)\}$$

The generating function for such partitions is

$$\frac{1+t+t^2+t^3+t^4+t^1t^4}{(1-t^5)(1-t^6)} = \frac{1+t+t^2+t^3+t^4+t^5}{(1-t^5)(1-t^6)} = \frac{1}{(1-t^1)(1-t^5)}$$

Example: Affine partitions of type F_4

Let $\mathcal{P}(F_4)$ be the family of colored partitions $(\lambda_1^{c_1} \le \cdots \le \lambda_k^{c_k})$ with parts

$$\lambda_i \in \{1, 2, \ldots, 16\} \cup \{5', 6', \ldots, 12'\}$$

such that each consecutive pair $(\lambda_i, \lambda_{i+1})$ is an *allowed pair*:

(16.16) (15,16) (15,15) (14,16) (14,15) (14,14) (13,16) (13,15) (13,14) (12,16) (12,15) (12,14) (12',16) (12',15) (12',14) (12',13) (11,16) (11,15) (11,14) (11,13) (11',16) (11',15) (11',14) (11',13) (11',12') (11',11') (10,16) (10,15) (10,14) (10,13) (10',16) (10',15) (10',14) (10',13) (10',12') (10',11') (9,16) (9,15) (9,14) (9,13) (9,12) (9',16) (9',15) (9',14) (9',13) (9',12') (9',11') (8,16) (8,15) (8,14) (8,13) (8,12') (8,11') (8',16) (8',15) (8',14) (8',13) (8',12) (7,16) (7,15) (7,14) (7,13) (7,12') (7,11') (7',16) (7',15) (7',14) (7',13) (7',12) (7',12') (7',11') (6,16) (6,15) (6,14) (6,13) (6,12') (6,11') (6',16) (6',15) (6',14) (6',13) (6',12) (6',12') (6',11') (5,16) (5,15) (5,14) (5,13) (5,12) (5,12') (5,11') (5',16) (5',15) (5',14) (5',13) (5',12) (5',12') (5',11) (5',11') (5',10') (4,16) (4,15) (4,14) (4,13) (4,12) (4,12') (4,11) (4,11') (4,10') (3,16) (3,15) (3,14) (3,13) (3,12) (3,12') (3,11) (3,11') (3,10') (2,16) (2,15) (2,14) (2,13) (2,12) (2,12') (2,11) (2,11') (2,10) (2,10') (2,9') (1,16) (1,15) (1,14) (1,13) (1,12) (1,12') (1,11) (1,11') (1,10) (1,10') (1,9) (1,9') (1,8) (1,7)

Weyl groups

A *Weyl group* W is a finite reflection group generated by simple reflections $S = \{s_1, \ldots, s_n\}$ with relations of the form

$$\begin{aligned} s_i^2 &= 1 \\ (s_i s_j)^{m_{ij}} &= 1 \text{ with } m_{ij} \in \{2,3,4,6\} \end{aligned}$$

The relations are encoded in a graph called a *Dynkin diagram* with vertices given by the generators and edges connecting s_i, s_j if $m_{ij} \geq 3$. A single edge implies $m_{ij} = 3$, a double edge implies $m_{ij} = 4$ and a triple edge implies $m_{ij} = 6$.

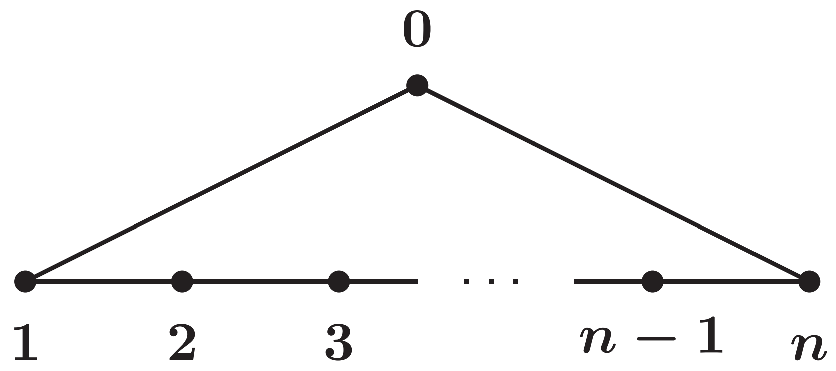
Example. The Symmetric Group S_{n+1} is a Weyl group with Dynkin diagram

Affine Weyl groups

For each finite Weyl group W there is an associated *affine Weyl group* $\widetilde{W}$ generated by $S \cup \{s_0\}$ with relations determined by the *affine Dynkin diagram*.

An expression $w = s_{i_1} s_{i_2} \ldots s_{i_p}$ is *reduced* if p is minimal over all expressions for w. The *length* of w is $\ell(w) = p$. The *Bruhat order* is a ranked partial order determined by the covering relation $v < w$ if $\ell(v) = \ell(w) - 1$ and $v = s_{i_1} \ldots \widehat{s_{i_j}} \ldots s_{i_p}$ for some j.

Example. The Affine Symmetric Group $\widetilde{S}_{n+1}$ has Dynkin diagram



Affine quotients $\widetilde{W}/W$

Defn. Let $\widetilde{W}^S$ be the minimal length coset representatives for $\widetilde{W}/W$. $\widetilde{W}^S$ inherits the length function and Bruhat order from $\widetilde{W}$.

Nice Properties. (Bott, Quillen, Garland-Raghunathan, Mitchell. . .)

1. The *affine Grassmannian* $\mathcal{L}_W$ is an infinite dimensional variety with the structure of a CW-complex indexed by $\widetilde{W}^S$

$$\mathcal{L}_W = \bigoplus_{w \in \widetilde{W}^S} e_w.$$

   where e_w is a cell of complex dimension $\ell(w)$.

2. $\mathcal{L}_W$ is homeomorphic to $\Omega_{\text{alg}} G$.

3. *Bott's Formula*: Let $e_1, \dots, e_n$ be the exponents of W. Then

$$\widetilde{W}^S(t) := \sum_{v \in \widetilde{W}^S} t^{\ell(v)} = \prod_{i=1}^{n} \frac{1}{(1 - t^{e_i})}$$

Example

$\tilde{C}_5$:

$$\begin{aligned}\widetilde{W}^S(t) &= \frac{1}{(1-t)(1-t^3)(1-t^5)(1-t^7)(1-t^9)} \\ &= 1 + t + t^2 + 2t^3 + 2t^4 + 3t^5 + 4t^6 + 5t^7 + 6t^8 + 8t^9 + \dots\end{aligned}$$

Affine Partitions and Bruhat Order

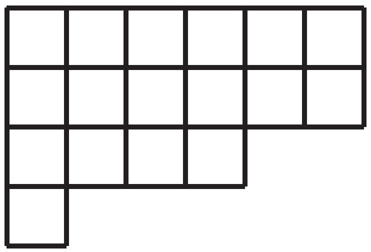
Generalizing work of Bousquet-Mélou-Eriksson (on Lecture Hall Partitions), Eriksson-Eriksson, Lam-Lapoint-Morse-Shimozono (k-core partitions), and Reiner . . .

Theorem. (Billey-Mitchell) For each type, there exists a family of colored partitions $\mathcal{P}$ in bijection with $\widetilde{W}^S$. Furthermore, if $\boldsymbol{shape}(\mu) \subset \boldsymbol{shape}(\lambda)$ in the generalized Young's lattice then $\mu \leq \lambda$ in Bruhat order on $\widetilde{W}^S$.

Defn. Let $r(\lambda)$ be the canonical reduced expression for λ obtained by removing the largest possible descent at each step going down to the identity.

Canonical Factorization. In type I Weyl groups, consider the factorization of $r(\lambda)$ into the largest segments ending with s_0:

$$r(\lambda) = s_0 \cdot s_3 s_2 s_1 s_0 \cdot s_5 s_4 s_3 s_2 s_1 s_0 \cdot s_5 s_4 s_3 s_2 s_1 s_0$$

$\lambda \mapsto$ 

Affine Partitions and Bruhat Order

In type C_n, the factors correspond with the finite set of minimal length coset representatives in $W/\langle s_2, \ldots, s_n \rangle$:

$$\Sigma(j) = \begin{cases} s_{j-1} \cdots s_2 s_1 s_0 & 1 \le j \le n+1 \\ s_{2n-j+1} \cdots s_{n-1} s_n s_{n-1} \cdots s_2 s_1 s_0 & n+1 < j \le 2n. \end{cases}$$

Claim. $\Sigma(i)\Sigma(j) \in \widetilde{W}^S \iff \begin{cases} i \le j & n < j \le 2n \\ i < j & 1 \le j \le n. \end{cases}$

Consequences.

1. If $r(\lambda) = \Sigma(j_1)\Sigma(j_2) \cdots \Sigma(j_k)$, then $j_1 \le j_2 \le \cdots \le j_k$. Furthermore, $j_i = j_{i+1}$ allowed only if $j_i > n$.

2. Let $\mathcal{P}(C_n) =$ partitions bounded by $2n-1$ with strictly increasing parts of size $1, \ldots, n$. Then the bijection between $\mathcal{P}(C_n)$ and $\widetilde{W}^S$ follow from the (known) identity

$$\frac{1}{(1-t)(1-t^3)(1-t^5)\ldots(1-t^{2n-1})} = \frac{(1+t)(1+t^2)\cdots(1+t^n)}{(1-t^{n+1})(1-t^{n+2})\cdots(1-t^{2n})}$$

Affine Schubert Varieties

Defn. For $\boldsymbol{\lambda} \in \widetilde{W}^S$, define the *affine Schubert variety*

$$X_\lambda = \overline{e_\lambda} = \bigoplus_{\mu \leq \lambda} e_\mu .$$

Defn. Let P_λ = Poincaré polynomial for $X_\lambda = \sum_{\mu \leq \lambda} t^{\ell(\mu)}$.

Examples. $\tilde{C}_5$:

$$\begin{aligned} P_{s_2 s_1 s_0} &= 1 + t + t^2 + t^3 && \textit{palindromic} \\ P_{s_0 s_2 s_1 s_0} &= 1 + t + t^2 + 2t^3 + t^4 && \textit{not palindromic} \end{aligned}$$

Affine Schubert Varieties

Questions. Which X_λ are smooth? Which X_λ are palindromic?

Implications.

$$\begin{aligned} X_\lambda \text{ smooth} \quad &\Longrightarrow \quad X_\lambda \text{ satisfies Poincaré duality integrally} \\ &\Longrightarrow \quad X_\lambda \text{ satisfies Poincaré duality rationally} \\ &\Longleftrightarrow \quad X_\lambda \text{ is palindromic by Carrell-Peterson theorem.} \end{aligned}$$

Obvious Candidates. X_λ is a *closed parabolic orbit*, if there exists a parabolic subgroup $P_I \subset \tilde{G}$ such that

$$X_\lambda = P_I \cdot e_{\mathrm{id}}.$$

These affine Schubert varieties are classical G_I/Q_I's, hence they are smooth.

Affine Partitions and Bruhat Order

Corollary(Billey-Mitchell). In type B_n, C_n, D_n, the palindromic elements in $\widetilde{W}^S$ are in bijection with staircase shapes with largest part at most n and the one row shapes. For these elements, the interval below in Bruhat order is isomporphic to the corresponding interval in Young's lattice.

Corollary. (Billey-Computer-Mitchell) In types $E_{6,7,8}, F_4, G_2$ the palindromics are indexed by strict partitions with at most 3 parts.

Two Main Theorems

Theorem. (Billey-Mitchell) X_λ is palindromic iff one of the following hold:

1. For all $i \in I$, $s_i\lambda < \lambda$. (closed parabolic orbit)
2. X_λ is a *chain*, i.e. $P_\lambda = \sum_{i=0}^{\ell(\lambda)} t^i$
3. Type $= B_3$ and $\lambda = s_3 s_2 s_0 \cdot s_3 s_2 s_1 \cdot s_3 s_2 s_0$.
4. Type $= A_n$ and X_λ is a *spiral variety*. or $X'_{k,n}$.

Defn. X_λ is a *spiral variety* in type A_n if λ has a single reduced expression and length kn for some k. (previously studied by Mitchell, Cohen-Lupercio-Segal)

Two Main Theorems

Theorem. (Billey-Mitchell) The following are equivalent:

1. X_λ is smooth.
2. X_λ satisfies Poincaré duality integrally.
3. X_λ closed parabolic orbit for some $I \subsetneq \tilde{S}$.
4. For all $i \in I$, $s_i\lambda < \lambda$.

Two Main Theorems

Corollary. In every type, there are only a finite number of nontrival smooth affine Schubert varieties corresponding with connected subgraphs of the Dynkin diagram containing s_0.

$$\begin{array}{llll} A_n: & \binom{n+1}{2} & E_{6,7,8}: & 10 \\ B_n: & 2n-2 & F_4: & 5 \\ C_n: & n & G_2: & 3 \\ D_n: & 2n & & \end{array}$$

Corollary. Smooth is equivalent to palindromic for all affine Schubert varieties of *non-cyclic simply laced* affine types only, i.e. $\tilde{D}, \tilde{E}$.

Corollary. Outside of type A, there are only finitely many palindromic X_λ.

Open Problems

1. Can affine partitions be extended to other Coxeter groups?
2. When does a family of colored partitions defined by allowed pairs have a simple generating function?
3. Is there a way to identify the elements in $\widetilde{W}^S$ with the set of partitions that correspond with Bott's formula.
4. Is there a family of partitions in bijection with $\widetilde{W}^S$ such that $v \leq w$ in Bruhat order if and only if the partition for v is contained in the partition for w; in analogy with k-core partions of Lam-Lapoint-Morse-Shimozono?
5. Do the segments have a nice interpretation in terms of homology/cohomology of X_λ?
6. Determine the singular locus of X_λ.