Enumeration of Parabolic Double Cosets in Symmetric Groups and Beyond

Sara Billey
University of Washington

Based on joint work with:
Matjaž Konvalinka, T. Kyle Petersen, William Slofstra and Bridget Tenner
FPSAC 2016 abstract

LSU Math Graduate Colloquium, February 29, 2016

Quote by Arnold Ross

"Think deeply of simple things"

Outline

Symmetric Groups

Notation.

- S_n is the group of permutations.
- $t_{i,j} = (i \leftrightarrow j) =$ transposition for $i < j$,
- $s_i = (i \leftrightarrow i+1) =$ simple transposition for $1 \leq i < n$.

Example. $w = [3, 4, 1, 2, 5] \in S_5$,

$$ws_4 = [3, 4, 1, 5, 2] \quad \text{and} \quad s_4 w = [3, 5, 1, 2, 4].$$

Symmetric Groups

Presentation.

S_n is generated by $s_1, s_2, \ldots, s_{n-1}$ with relations

$$
\begin{aligned}
&s_i s_i = 1 \\
&(s_i s_j)^2 = 1 \text{ if } |i-j| > 1 \\
&(s_i s_{i+1})^3 = 1
\end{aligned}
$$

This presentation of S_n by generators and relations is encoded an edge labeled chain, called a Coxeter graph.

$$S_7 \approx \quad \bullet_1 \overset{3}{\text{---}} \bullet_2 \overset{3}{\text{---}} \bullet_3 \overset{3}{\text{---}} \bullet_4 \overset{3}{\text{---}} \bullet_5 \overset{3}{\text{---}} \bullet_6$$

Symmetric Groups

Notation. Given any $w \in S_n$ write

$$w = s_{i_1} s_{i_2} \cdots s_{i_k}$$

in a minimal number of generators. Then

- k is the length of w denoted $\ell(w)$.
- $\ell(w) = \#\{(i < j) \mid w(i) > w(j)\}$ (inversions).
- $s_{i_1} s_{i_2} \cdots s_{i_k}$ is a reduced expression for w.

Example. $w = [2, 1, 4, 3, 7, 6, 5] \in S_7$ has 5 inversions, $\ell(w) = 5$.

$$w = [2, 1, 4, 3, 7, 6, 5] = s_1 s_3 s_6 s_5 s_6 = s_3 s_1 s_6 s_5 s_6 = s_3 s_1 s_5 s_6 s_5 = \dots$$

Symmetric Groups

Poincaré polynomials. Interesting q-analog of $n!$:

$$\sum_{w \in S_n} q^{\ell(w)} = (1+q)(1+q+q^2)\cdots(1+q+q^2+\ldots+q^{n-1}) = [n]_q!.$$

Examples.

$$[2]_q! = 1 + q$$
$$[3]_q! = 1 + 2q + 2q^2 + q^3$$
$$[4]_q! = 1 + 3q + 5q^2 + 6q^3 + 5q^4 + 3q^5 + q^6$$

Symmetric Groups

Poincaré polynomials. Interesting q-analog of $n!$:

$$\sum_{w \in S_n} q^{\ell(w)} = (1+q)(1+q+q^2)\cdots(1+q+q^2+\ldots+q^{n-1}) = [n]_q!.$$

Examples.

$$[2]_q! = 1 + q$$
$$[3]_q! = 1 + 2q + 2q^2 + q^3$$
$$[4]_q! = 1 + 3q + 5q^2 + 6q^3 + 5q^4 + 3q^5 + q^6$$

Open. Find a simple formula for the coefficient of q^k in $[n]_q!$

Symmetric Groups

Eulerian polynomials. Another interesting q-analog of $n!$:

$$A_n(q) = \sum_{k=0}^{n-1} A_{n,k} q^k = \sum_{w \in S_n} q^{asc(w)}$$

where $Ascents(w) = \{i \mid w(i) > w(i+1)\}$ and $asc(w) = \#Ascents(w)$. See Petersen's book "Eulerian Numbers."

Examples.

$$A_2(q) = 1 + q$$
$$A_3(q) = 1 + 4q + q^2$$
$$A_4(q) = 1 + 11q + 11q^2 + q^3$$

Symmetric Groups

Eulerian polynomials. Another interesting q-analog of $n!$:

$$A_n(q) = \sum_{k=0}^{n-1} A_{n,k} q^k = \sum_{w \in S_n} q^{asc(w)}$$

where $Ascents(w) = \{i \mid w(i) > w(i+1)\}$ and $asc(w) = \#Ascents(w)$. See Petersen's book "Eulerian Numbers."

Examples.

$$A_2(q) = 1 + q$$
$$A_3(q) = 1 + 4q + q^2$$
$$A_4(q) = 1 + 11q + 11q^2 + q^3$$

Theorem. (Holte 1997, Diaconis-Fulman 2009) When adding together n large randomly chosen numbers in any base, the probability of carrying a k for $0 \leq k < n$ is approximately $A_{n,k}/n!$.

Parabolic Subgroups and Cosets

Defn. For any subset $I \in \{1, 2, \ldots, n-1\} = [n-1]$, let W_I be the parabolic subgroup of S_n generated by $\langle s_i \mid i \in I \rangle$.

Defn. Sets of permutations of the form wW_I (or $W_I w$) are left (or right) parabolic cosets for W_I for any $w \in S_n$.

Example. Take $I = \{1, 3, 4\}$ and $w = [3, 4, 1, 2, 5]$. Then the left coset wW_I includes the 12 permutations

[34125]	[34152]	[34215]	[34512]	[34251]	[34521]
[43125]	[43152]	[43215]	[43512]	[43251]	[43521]

Parabolic Subgroups and Cosets

Defn. For any subset $I \in \{1, 2, \ldots, n-1\} = [n-1]$, let W_I be the parabolic subgroup of S_n generated by $\langle s_i \mid i \in I \rangle$.

Defn. Sets of permutations of the form wW_I (or $W_I w$) are left (or right) parabolic cosets for W_I for any $w \in S_n$.

Example. Take $I = \{1, 3, 4\}$ and $w = [3, 4, 1, 2, 5]$. Then the left coset wW_I includes the 12 permutations

$$\begin{array}{cccccc} [34125] & [34152] & [34215] & [34512] & [34251] & [34521] \\ [43125] & [43152] & [43215] & [43512] & [43251] & [43521] \end{array}$$

Facts.

- Every parabolic coset has a unique minimal and a unique maximal length element.
- Every parabolic coset for W_I has size $|W_I|$.
- S_n is the disjoint union of the $n!/|W_I|$ left parabolic cosets S_n/W_I.

Parabolic Double Cosets

Defn. Let $I, J \in [n-1]$ and $w \in S_n$, then the sets of permutations the form $W_I \cdot w \cdot W_J$ are parabolic double cosets.

Example. Take $I = \{2\},\ J = \{1, 3, 4\}$ and $w = [3, 4, 1, 2, 5]$. Then the parabolic double coset $W_I w W_J$ includes

Parabolic Double Cosets

Defn. Let $I, J \in [n-1]$ and $w \in S_n$, then the sets of permutations the form $W_I \cdot w \cdot W_J$ are parabolic double cosets.

Example. Take $I = \{2\},\ J = \{1,3,4\}$ and $w = [3,4,1,2,5]$. Then the parabolic double coset $W_I w W_J$ includes

[34125]	[34152]	[34215]	[34512]	[34251]	[34521]
[43125]	[43152]	[43215]	[43512]	[43251]	[43521]

plus

[24135]	[24153]	[24315]	[24513]	[24351]	[24531]
[42135]	[42153]	[42315]	[42513]	[42351]	[42531]

Parabolic Double Cosets

Defn. Let $I, J \in [n-1]$ and $w \in S_n$, then the sets of permutations the form $W_I \cdot w \cdot W_J$ are parabolic double cosets.

Example. Take $I = \{2\}, \; J = \{1,3,4\}$ and $w = [3,4,1,2,5]$. Then the parabolic double coset $W_I w W_J$ includes

$$\begin{array}{cccccc} [34125] & [34152] & [34215] & [34512] & [34251] & [34521] \\ [43125] & [43152] & [43215] & [43512] & [43251] & [43521] \end{array}$$

plus

$$\begin{array}{cccccc} [24135] & [24153] & [24315] & [24513] & [24351] & [24531] \\ [42135] & [42153] & [42315] & [42513] & [42351] & [42531] \end{array}$$

Example. $W_I \; [4,5,1,2,3] \; W_J$ has 12 elements.

Parabolic Double Cosets

Facts.

- Parabolic double coset for W_I, W_J can have different sizes.
- S_n is the disjoint union of the parabolic double cosets

$$W_I \backslash S_n / W_J = \{W_I w W_J \mid w \in S_n\}.$$

- Every parabolic double coset has a unique minimal and a unique maximal length element.

Parabolic Double Cosets

Facts.

- Parabolic double coset for W_I, W_J can have different sizes.
- S_n is the disjoint union of the parabolic double cosets

$$W_I \backslash S_n / W_J = \{W_I w W_J \mid w \in S_n\}.$$

- Every parabolic double coset has a unique minimal and a unique maximal length element.

Thm.(Kobayashi 2011) Every parabolic double coset is an interval in Bruhat order. The follow polynomials are palindromic

$$P_{I,w,J}(q) = \sum_{v \in W_I w W_J} q^{\ell(v)}.$$

Connection to Richardson Varieties

Thm. The Richardson variety in $GL_n(\mathbb{C})/B$ indexed by $u < v$ is smooth if and only if the following polynomial is palindromic

$$\sum_{u \leq v \leq w} q^{\ell(v)}.$$

References on smooth Richardson varieties: See book by Billey-Lakshmibai, and papers by Carrell, Billey-Coskun, Lam-Knutson-Speyer, Kreiman-Lakshmibai, Knutson-Woo-Yong, Lenagan-Yakimov and many more.

Counting Parabolic Double Cosets

Question 1. For a fixed I, J, how many distinct parabolic double cosets are there in $W_I \backslash S_n / W_J$?

Question 2. Is there a formula for $f(n) = \sum_{I,J} |W_I \backslash S_n / W_J|$?

Question 3. How many distinct parabolic double cosets are there in S_n in total?

Counting Double Cosets

- G= finite group
- H, K = subgroups of G
- $H\backslash G/K$ = *double cosets* of G with respect to H, K
  $= \{HgK : g \in G\}$

Generlization of Question 1. What is the size of $H\backslash G/K$?

Counting Double Cosets

- G= finite group
- H, K = subgroups of G
- $H\backslash G/K$ = *double cosets* of G with respect to H, K
  $= \{HgK : \ g \in G\}$

Generlization of Question 1. What is the size of $H\backslash G/K$?

One Answer..
The size of $H\backslash G/K$ is given by the inner product of the characters of the two trivial representations on H and K respectively induced up to G.

Reference: Stanley's "Enumerative Combinatorics" Ex 7.77a.

Counting Parabolic Double Cosets

Question 2. Is there a formula for $f(n) = \sum_{I,J} |W_I \backslash S_n / W_J|$?

Data. $1, 1, 5, 33, 281, 2961, 37277, 546193, 9132865, 171634161$ (A120733 in OEIS)

This counts the number of "two-way contingency tables" (see Diaconis-Gangoli 1994), the dimensions of the graded components of the Hopf algebra MQSym (see Duchamp-Hivert-Thibon 2002), and the number of cells in a two-sided analogue of the Coxeter complex (Petersen).

Counting Parabolic Double Cosets

Question 3. How many distinct parabolic double cosets are there in S_n in total?

Data.: $p(n) = |\{W_I v W_J \mid v \in S_n, I, J \subset [n-1]\}|$,

$1, 3, 19, 167, 1791, 22715, 334031, 5597524, 105351108, 2200768698$

Not formerly in the OEIS! Now, see A260700.

Counting Parabolic Double Cosets

Question 3. How many distinct parabolic double cosets are there in S_n in total?

Defn. For $w \in S_n$, let c_w be the number of distinct parabolic double cosets with w minimal.

One Answer. $p(n) = \sum_{w \in S_n} c_w$.

Representing Parabolic Double Cosets

Lemma. w is minimal in $W_I w W_J$ if and only if $\ell(s_i w) > \ell(w)$ for all $i \in I$ and $\ell(w s_j) > \ell(w)$ for all $j \in J$. So

$$c_w = \#\{W_I w W_J \mid I \subset Ascent(w^{-1}), J \subset Ascent(w)\}.$$

Observation. Sometimes $W_I w W_J = W_{I'} w W_{J'}$ even if $I, I' \subset Ascent(w^{-1})$ and $J, J' \subset Ascent(w)$.

Dilemma. Which representation is best for enumeration?

Representing Parabolic Double Cosets

Example. $w = [3,4,1,2,5] = w^{-1}$, $Ascent(w) =$

Representing Parabolic Double Cosets

Example. $w = [3, 4, 1, 2, 5] = w^{-1}$, $Ascent(w) = \{1, 3, 4\}$,

Representing Parabolic Double Cosets

Example. $w = [3, 4, 1, 2, 5] = w^{-1}$, $Ascent(w) = \{1, 3, 4\}$, $ws_1 = [4, 3, 1, 2, 5] = s_3 w$ so $W_{\{3\}} w W_{\{\}} = W_{\{\}} w W_{\{1\}}$.

Representing Parabolic Double Cosets

Example. $w = [3,4,1,2,5] = w^{-1}$, $Ascent(w) = \{1,3,4\}$,
$ws_1 = [4,3,1,2,5] = s_3 w$ so $W_{\{3\}} w W_{\{\}} = W_{\{\}} w W_{\{1\}}$.
$ws_4 = [3,4,1,5,2] \neq s_i w$ for any i and
$s_4 w = [3,5,1,2,4] \neq ws_i$ for any i.

Representing Parabolic Double Cosets

Example. $w = [3,4,1,2,5] = w^{-1}$, $Ascent(w) = \{1,3,4\}$,
$ws_1 = [4,3,1,2,5] = s_3 w$ so $W_{\{3\}} w W_{\{\}} = W_{\{\}} w W_{\{1\}}$.
$ws_4 = [3,4,1,5,2] \neq s_i w$ for any i and
$s_4 w = [3,5,1,2,4] \neq ws_i$ for any i.

Defn. A small ascent for w is an ascent j such that $ws_j = s_i w$.
Every other ascent is large.

Representing Parabolic Double Cosets

Example. $w = [3, 4, 1, 2, 5] = w^{-1}$, $Ascent(w) = \{1, 3, 4\}$,
$ws_1 = [4, 3, 1, 2, 5] = s_3 w$ so $W_{\{3\}} w W_{\{\}} = W_{\{\}} w W_{\{1\}}$.
$ws_4 = [3, 4, 1, 5, 2] \neq s_i w$ for any i and
$s_4 w = [3, 5, 1, 2, 4] \neq ws_i$ for any i.

Defn. A small ascent for w is an ascent j such that $ws_j = s_i w$.
Every other ascent is large.

Enumeration Principle. To count distinct parabolic double cosets $W_I w W_J$ with w minimal, J can contain any subset of large ascents for w, I can contain any subset of large ascents for w^{-1}, count the small ascents very carefully!

Counting Parabolic Double Cosets

Theorem. (Billey-Konvalinka-Petersen-Slofstra-Tenner)

1. There is a finite family of 81 integer sequences $\{b_m^{\mathcal{I}} \mid m \geq 0\}$, such that for any permutation w, the total number of parabolic double cosets with minimal element w is equal to

$$c_w = 2^{|\operatorname{Floats}(w)|} \sum_{T \subseteq \operatorname{Tethers}(w)} \left(\prod_{R \in \operatorname{Rafts}(w)} b_{|R|}^{\mathcal{I}(R,T)} \right).$$

Counting Parabolic Double Cosets

Theorem. (Billey-Konvalinka-Petersen-Slofstra-Tenner)

1. There is a finite family of 81 integer sequences $\{b_m^{\mathcal{I}} \mid m \geq 0\}$, such that for any permutation w, the total number of parabolic double cosets with minimal element w is equal to

$$c_w = 2^{|\,\mathrm{Floats}(w)|} \sum_{T \subseteq \mathrm{Tethers}(w)} \left(\prod_{R \in \mathrm{Rafts}(w)} b_{|R|}^{\mathcal{I}(R,T)} \right).$$

2. The sequences $b_m^{\mathcal{I}}$ satisfy a linear recurrence, and thus can be easily computed in time linear in m.

Counting Parabolic Double Cosets

Theorem. (Billey-Konvalinka-Petersen-Slofstra-Tenner)

1. There is a finite family of 81 integer sequences $\{b_m^{\mathcal{I}} \mid m \geq 0\}$, such that for any permutation w, the total number of parabolic double cosets with minimal element w is equal to

$$c_w = 2^{|\operatorname{Floats}(w)|} \sum_{T \subseteq \operatorname{Tethers}(w)} \left(\prod_{R \in \operatorname{Rafts}(w)} b_{|R|}^{\mathcal{I}(R,T)} \right).$$

2. The sequences $b_m^{\mathcal{I}}$ satisfy a linear recurrence, and thus can be easily computed in time linear in m.
3. The expected number of tethers for any given permutation is approximately $1/n$.

The Marine Model

Main Formula. For $w \in S_n$,

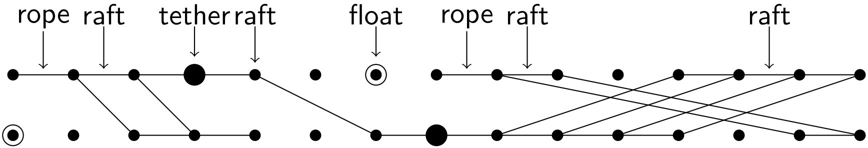
$$c_w = 2^{|\,\mathrm{Floats}(w)|} \sum_{T \subseteq \mathrm{Tethers}(w)} \left(\prod_{R \in \mathrm{Rafts}(w)} b_{|R|}^{\mathcal{I}(R,T)} \right).$$

The w-Ocean.

1. Take 2 parallel copies of the Coxeter graph G of S_n
2. Connect vertex $i \in Ascent(w^{-1})$ and vertex $j \in Ascent(w)$ by a new edge called planks whenever $ws_j = s_i w$.
3. Remove all edges not incident to a small ascent.

The Marine Model

Example. Rafts, tethers, floats and ropes of the w ocean
$w = (1, 3, 4, 5, 7, 8, 2, 6, 14, 15, 16, 9, 10, 11, 12, 13)$.

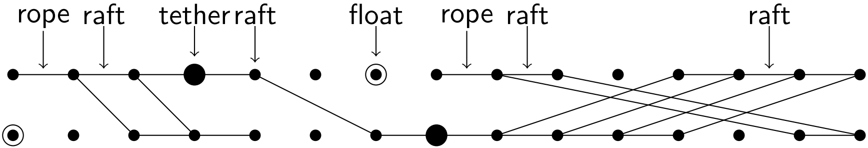


The Marine Model Terminology.

1. Raft – a maximal connected component of adjacent planks.
2. Float – a large ascent not adjacent to any rafts.
3. Rope – a large ascent adjacent to exactly one raft.
4. Tether – a large ascent connected to two rafts.

The Marine Model

Example. $w = (1, 3, 4, 5, 7, 8, 2, 6, 14, 15, 16, 9, 10, 11, 12, 13)$.



$$c_w = 2^{|\,\text{Floats}(w)|} \sum_{T \subseteq \text{Tethers}(w)} \left(\prod_{R \in \text{Rafts}(w)} b_{|R|}^{\mathcal{I}(R,T)} \right).$$

$$= 2^2 \big(b_2^{(4,8)} \cdot b_1^{(4,8)} \cdot b_2^{(4,8)} \cdot b_4^{(4,8)} + b_2^{(4)} \cdot b_1^{(4)} \cdot b_2^{(4)} \cdot b_4^{(4)}$$

$$+ b_2^{(8)} \cdot b_1^{(8)} \cdot b_2^{(8)} \cdot b_4^{(8)} + b_2^{()} \cdot b_1^{()} \cdot b_2^{()} \cdot b_4^{()} \big)$$

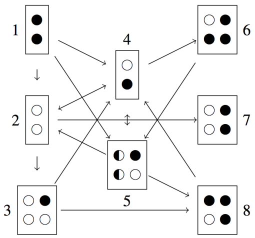
$$= 2^2 (71280 + 136620 + 144180 + 245640) = 2,390,880$$

Proof Sketch

Defn. (I, J) is lex minimal over all pairs (I', J') such that $D = W'_I w W'_J$ provided $|I| < |I'|$ or $|I| = |I'|$ and $|J| < |J'|$.

Lemma. The lex minimal pair for a parabolic double coset is unique.

Lemma. Lex minimal pairs along any one raft correspond with words in the finite automaton below (loops are omitted), hence then are enumerated by a rational generating function $P^{\mathcal{I}}(x)/Q(x)$ by the Transfer Matrix Method.



Coxeter Groups

- G = *Coxeter graph* with vertices $\{1, 2, \ldots, n\}$,
  edges labeled by $\mathbb{Z}_{\geq 3} \cup \infty$.

$$\bullet_1 \overset{4}{\text{——}} \bullet_2 \overset{3}{\text{——}} \bullet_3 \overset{3}{\text{——}} \bullet_4 \quad \approx \quad \bullet_1 \overset{4}{\text{——}} \bullet_2 \text{——} \bullet_3 \text{——} \bullet_4$$

- W = *Coxeter group* generated by $S = \{s_1, s_2, \ldots, s_n\}$ with relations
  1. $s_i^2 = 1$.
  2. $s_i s_j = s_j s_i$ if i, j not adjacent in G.
  3. $\underbrace{s_i s_j s_i \cdots}_{m(i,j) \text{ gens}} = \underbrace{s_j s_i s_j \cdots}_{m(i,j) \text{ gens}}$ if i, j connected by edge labeled $m(i, j)$.

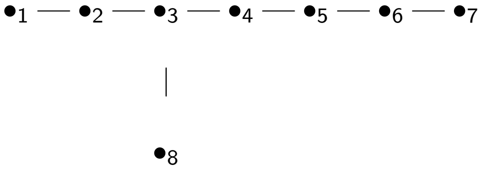
Examples

Dihedral groups: Dih_{10} $\qquad \bullet_1 \overset{5}{\text{———}} \bullet_2$

Symmetric groups: S_5 $\qquad \bullet_1 \text{ — } \bullet_2 \text{ — } \bullet_3 \text{ — } \bullet_4$

Hyperoctahedral groups: B_4 $\qquad \bullet_1 \overset{4}{\text{———}} \bullet_2 \text{ — } \bullet_3 \text{ — } \bullet_4$

E_8:



Generalizing the notation from Symmetric Groups

- $W =$ *Coxeter group* generated by $S = \{s_1, s_2, \ldots, s_n\}$ with special relations.
- $\ell(w) =$ *length* of $w =$ length of a reduced expression for w.
- $W_I = \langle s_i \mid i \in I \rangle$ is a parabolic subgroup of W.
- $W_I w W_J$ is a parabolic double coset of W for any $I, J \subset [n]$, $w \in W$.
- $c_w =$ number of distinct parabolic double cosets in W with minimal element w.

Generalizing Main Theorem to Coxeter Groups

Theorem. (Billey-Konvalinka-Petersen-Slofstra-Tenner)

1. For every finite Coxeter group W and $w \in W$, we have

$$c_w = 2^{|\,\mathrm{Floats}(w)|} \sum_{\substack{T \subseteq \mathrm{Tethers}(w) \\ W \subseteq \mathrm{Wharfs}(w)}} \left(\prod_{R \in \mathrm{Rafts}(w)} b_{|R|}^{\mathcal{I}(R,T,W)} \right).$$

Generalizing Main Theorem to Coxeter Groups

Theorem. (Billey-Konvalinka-Petersen-Slofstra-Tenner)

1. For every finite Coxeter group W and $w \in W$, we have

$$c_w = 2^{|\operatorname{Floats}(w)|} \sum_{\substack{T \subseteq \operatorname{Tethers}(w) \\ W \subseteq \operatorname{Wharfs}(w)}} \left(\prod_{R \in \operatorname{Rafts}(w)} b_{|R|}^{\mathcal{I}(R,T,W)} \right).$$

2. The sequences $b_m^{\mathcal{I}(R,T,W)}$ satisfy a linear recurrence.

Generalizing Main Theorem to Coxeter Groups

Theorem. (Billey-Konvalinka-Petersen-Slofstra-Tenner)

1. For every finite Coxeter group W and $w \in W$, we have

$$c_w = 2^{|\,\mathrm{Floats}(w)|} \sum_{\substack{T \subseteq \mathrm{Tethers}(w) \\ W \subseteq \mathrm{Wharfs}(w)}} \left(\prod_{R \in \mathrm{Rafts}(w)} b_{|R|}^{\mathcal{I}(R,T,W)} \right).$$

2. The sequences $b_m^{\mathcal{I}(R,T,W)}$ satisfy a linear recurrence.
3. We generalize the formula for c_w to infinite families of Coxeter groups given by subdividing a fixed Coxeter graph G.

Mozes Numbers Game

Algorithm. Generates canonical representative for each element in a Coxeter group using its graph.
(See Mozes 1990, Eriksson-Eriksson 1998, Björner-Brenti Book)

Input: Coxeter graph G and expression $s_{i_1} s_{i_2} \ldots s_{i_p} = w$.

Start: Each vertex of graph G assigned value 1. Replace each edge (i,j) of G by two opposing directed edges labeled $f_{ij} > 0$ and $f_{ji} > 0$ so that $f_{ij} f_{ji} = 4\cos^2\left(\frac{\pi}{m(i,j)}\right)$ or $f_{ij} f_{ji} = 4$ if $m(i,j) = \infty$.

Good choices:

$m(i,j)$	f_{ij}	f_{ji}
3	1	1
4	2	1
6	3	1

Mozes Numbers Game

Loop. For each s_{i_k} in $s_{i_1} s_{i_2} \ldots s_{i_p}$ *fire node i_k*.

To fire node i, add to the value of each neighbor j the current value at node i multiplied by f_{ij}. Negate the value on node i.

Output.: $G(w) =$ the final values on the nodes of G.

Mozes Numbers Game

Loop. For each s_{i_k} in $s_{i_1}s_{i_2}\dots s_{i_p}$ *fire node i_k*.

To fire node i, add to the value of each neighbor j the current value at node i multiplied by f_{ij}. Negate the value on node i.

Output.: $G(w) =$ the final values on the nodes of G.

Properties:

1. Output only depends on the product $s_{i_1}s_{i_2}\dots s_{i_p}$ and not on the particular choice of expression.
2. Node i is negative in $G(w)$ iff $\ell(ws_i) < \ell(w)$.
3. Node i never has value 0.
4. If $I \subset S$, modify the game to get representatives for W/W_I by starting with initial value 0 on nodes in I. Then $ws_i = w$ iff node i has value 0. Useful for studying parabolic cosets.

Open Problems

1. Follow up to Question 3: Is there a simpler or more efficient formula for the total number of distinct parabolic double cosets are there in S_n than the one given here?

2. Follow up to Question 2: Is there a simpler or more efficient formula for $f(n) = \sum_{I,J} |W_I \backslash S_n / W_J|$?

3. What other families of double cosets have interesting enumeration formulas?