Grassmannians and Other Schubert Varieties

Sara Billey
University of Washington
http://www.math.washington.edu/∼billey

What is Algebraic Geometry?
April 14, 2007

Famous Quotations

Arnold Ross (and PROMYS).

"Think deeply of simple things."

Angela Gibney. Why do algebraic geometers love moduli spaces?

"It is just like with people, if you want to get to know someone, go to their family reunion."

Goal. Focus our attention on a particular family of varieties which are indexed by combinatorial data where lots is known about their structure and yet lots is still open.

Schubert Varieties

A *Schubert variety* is a member of a family of projective varieties whose points are indexed by matrices and whose defining equations are determinantal minors.

Typical properties:

- This family of varieties is indexed by combinatorial objects; e.g. partitions, permutations, or Weyl group elements.
- Some are smooth and some are singular.
- Their cohomology rings include familiar subsets and quotients of polynomial rings including the symmetric functions.

Enumerative Geometry

Approximately 150 years ago... Grassmann, Schubert, Pieri, Giambelli, Severi, and others began the study of *enumerative geometry*.

Early questions:

- What is the dimension of the intersection between two general lines in $\mathbb{R}^2$?
- How many lines intersect two given lines and a given point in $\mathbb{R}^3$?
- How many lines intersect four given lines in $\mathbb{R}^3$?

Modern questions:

- How many points are in the intersection of 2,3,4,... Schubert varieties in general position?

Why Study Schubert Varieties?

1. It can be useful to see points, lines, planes etc as families with certain properties.

2. Schubert varieties provide interesting examples for test cases and future research in algebraic geometry.

3. Applications in discrete geometry, computer graphics, and computer vision.

Outline

1. Review of vector spaces and projective spaces
2. Introduction to Grassmannians, Flag Manifolds and Schubert Varieties
3. Five Fun Facts on Schubert Varieties
4. Permutation Arrays = higher dimensional analog of permutation matrices
5. Solving Schubert problems with Permutation Arrays

Vector Spaces

- V is a *vector space* over a field $\mathbb{F}$ if it is closed under addition and multiplication by scalars in $\mathbb{F}$.

- $B = \{b_1, \ldots, b_k\}$ is a *basis* for V if for every $a \in V$ there exist unique scalars $c_1, \ldots, c_k \in \mathbb{F}$ such that

$$a = c_1 b_1 + c_2 b_2 + \cdots + c_k b_k = (c_1, c_2, \ldots, c_k) \in \mathbb{F}^k.$$

- The *dimension* of V equals the size of a basis.

- A *subspace* U of a vector space V is any subset of the vectors in V that is closed under addition and scalar multiplication.

Fact. Any basis for U can be extended to a basis for V.

Projective Spaces

Defn.

- $\mathbb{P}(V) \ = \ \{$1-dim subspaces of $V\} \ = \ \frac{V}{\langle d \cdot a = a \rangle}$.

1-dim subspaces	$=$	lines in V through 0	$\longleftrightarrow$	points in $\mathbb{P}(V)$
2-dim subspaces	$=$	planes in V through 0	$\longleftrightarrow$	lines in $\mathbb{P}(V)$

- Given a basis $B = \{b_1, b_2, \ldots, b_k\}$ for V, the line spanned by $a = c_1 b_1 + c_2 b_2 + \cdots + c_k b_k \in \mathbb{P}(V)$ has *homogeneous coordinates*

$$[c_1 : c_2 : \cdots : c_k] = [dc_1 : dc_2 : \cdots : dc_k] \quad \text{for any } d \in \mathbb{F}.$$

The Grassmannian Varieties

Definition. Fix a vector space V over $\mathbb{C}$ (or $\mathbb{R}$, $\mathbb{Z}_2$,...) with basis $B = \{e_1, \ldots, e_n\}$. The *Grassmannian variety*

$$G(k, n) = \{k\text{-dimensional subspaces of } V\}.$$

Question.

How can we impose the structure of a variety or a manifold on this set?

The Grassmannian Varieties

Answer. Relate $G(k,n)$ to the vector space of $k \times n$ matrices.

$$U = \text{span}\langle 6e_1 + 3e_2, \quad 4e_1 + 2e_3, \quad 9e_1 + e_3 + e_4 \rangle \in G(3,4)$$

$$M_U = \begin{bmatrix} 6 & 3 & 0 & 0 \\ 4 & 0 & 2 & 0 \\ 9 & 0 & 1 & 1 \end{bmatrix}$$

- $U \in G(k,n) \iff$ rows of M_U are independent vectors in $V \iff$ some $k \times k$ minor of M_U is NOT zero.

Plücker Coordinates

- Define $f_{j_1,j_2,\ldots,j_k}$ to be the polynomial given by the determinant of the matrix
$$\begin{bmatrix} x_{1,j_1} & x_{1,j_2} & \ldots & x_{1,j_k} \\ x_{2,j_1} & x_{2,j_2} & \ldots & x_{2,j_k} \\ \vdots & \vdots & \vdots & \vdots \\ x_{kj_1} & x_{kj_2} & \ldots & x_{kj_k} \end{bmatrix}$$

- $G(k,n)$ is an open set in the Zariski topology on $k \times n$ matrices defined as the complement of the union over all k-subsets of $\{1,2,\ldots,n\}$ of the varieties $V(f_{j_1,j_2,\ldots,j_k})$.

- All the determinants $f_{j_1,j_2,\ldots,j_k}$ are homogeneous polynomials of degree k so $G(k,n)$ can be thought of as an open set in projective space.

The Grassmannian Varieties

Canonical Form. Every subspace in $G(k,n)$ can be represented by a unique matrix in row echelon form.

Example.

$$U = \text{span}\langle 6e_1 + 3e_2, \quad 4e_1 + 2e_3, \quad 9e_1 + e_3 + e_4 \rangle \in G(3,4)$$

$$\approx \begin{bmatrix} 6 & 3 & 0 & 0 \\ 4 & 0 & 2 & 0 \\ 9 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 7 & 0 & 0 & 1 \end{bmatrix}$$

$$\approx \langle 2e_1 + e_2, \quad 2e_1 + e_3, \quad 7e_1 + e_4 \rangle$$

Subspaces and Subsets

Example.

$$\boldsymbol{U} = \mathbf{RowSpan}\begin{bmatrix} \mathbf{5} & \mathbf{9} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{5} & \mathbf{8} & \mathbf{0} & \mathbf{9} & \mathbf{7} & \mathbf{9} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{4} & \mathbf{6} & \mathbf{0} & \mathbf{2} & \mathbf{6} & \mathbf{4} & \mathbf{0} & \mathbf{3} & \mathbf{1} & \mathbf{0} \end{bmatrix} \in \boldsymbol{G}(\mathbf{3}, \mathbf{10}).$$

$$\mathbf{position}(\boldsymbol{U}) = \{\mathbf{3}, \mathbf{7}, \mathbf{9}\}$$

Definition.

If $\boldsymbol{U} \in \boldsymbol{G}(\boldsymbol{k}, \boldsymbol{n})$ and $\boldsymbol{M_U}$ is the corresponding matrix in canonical form then the columns of the leading 1's of the rows of $\boldsymbol{M_U}$ determine a subset of size $\boldsymbol{k}$ in $\{\mathbf{1}, \mathbf{2}, \ldots, \boldsymbol{n}\} := [\boldsymbol{n}]$. There are 0's to the right of each leading 1 and 0's above and below each leading 1. This $\boldsymbol{k}$-subset determines the *position* of $\boldsymbol{U}$ with respect to the fixed basis.

The Schubert Cell C_{j} in $G(k,n)$

Defn. Let $\mathbf{j} = \{j_1 < j_2 < \cdots < j_k\} \in [n]$. A *Schubert cell* is

$$C_{\mathrm{j}} = \{U \in G(k,n) \mid \text{position}(U) = \{j_1, \ldots, j_k\}\}$$

Fact. $G(k,n) = \bigcup C_{\mathrm{j}}$ over all k-subsets of $[n]$

Example. In $G(3,10)$,

$$C_{\{3,7,9\}} = \left\{ \begin{bmatrix} * & * & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & * & * & * & 1 & 0 & 0 & 0 \\ * & * & 0 & * & * & * & 0 & * & 1 & 0 \end{bmatrix} \right\}$$

- Observe $\dim(C_{\{3,7,9\}}) = 13$.
- In general, $\dim(C_{\mathrm{j}}) = \sum j_i - i$.

Schubert Varieties in $G(k, n)$

Defn. Let $\mathbf{j} = \{j_1 < j_2 < \cdots < j_k\} \in [n]$. A *Schubert variety* is

$$X_{\mathbf{j}} = \text{Closure of } C_{\mathbf{j}} \text{ under Zariski topology.}$$

Question. In $G(3, 10)$, what polynomials vanish on $C_{\{3,7,9\}}$?

$$C_{\{3,7,9\}} = \left\{ \begin{bmatrix} * & * & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & * & * & * & 1 & 0 & 0 & 0 \\ * & * & 0 & * & * & * & 0 & * & 1 & 0 \end{bmatrix} \right\}$$

Answer. All minors f_{j_1,j_2,j_3} with $\left\{ \begin{array}{c} 4 \leq j_1 \leq 8 \\ \text{or } j_1 = 3 \text{ and } 8 \leq j_2 \leq 9 \\ \text{or } j_1 = 3, j_2 = 7 \text{ and } j_3 = 10 \end{array} \right\}$

In other words, the canonical form for any subspace in $\overline{C_{\mathbf{j}}}$ has 0's to the right of column j_i in each row i.

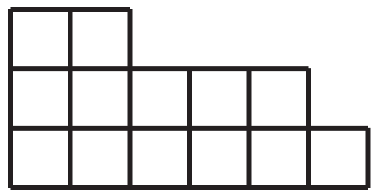
k-Subsets and Partitions

Defn. A *partition* of a number n is a weakly increasing sequence of non-negative integers

$$\lambda = (\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k)$$

such that $n = \sum \lambda_i = |\lambda|$.

Partitions can be visualized by their *Ferrers diagram*

$(2, 5, 6) \longrightarrow$ 

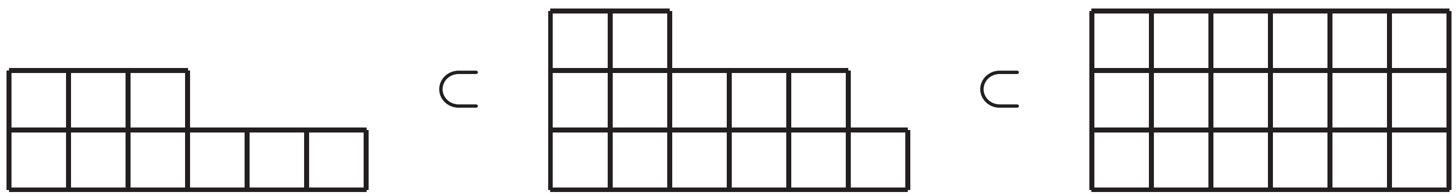
Fact. There is a bijection between k-subsets of $\{1, 2, \ldots, n\}$ and partitions whose Ferrers diagram is contained in the $k \times (n-k)$ rectangle given by

$$\mathbf{shape} : \{j_1 < \ldots < j_k\} \mapsto (j_1 - 1, j_2 - 2, \ldots, j_k - k).$$

A Poset on Partitions

Defn. A *partial order* or a *poset* is a reflexive, anti-symmetric, and transitive relation on a set.

Defn. *Young's Lattice*
If $\lambda = (\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k)$ and $\mu = (\mu_1 \leq \mu_2 \leq \cdots \leq \mu_k)$ then $\lambda \subset \mu$ if the Ferrers diagram for λ fits inside the Ferrers diagram for μ.



Facts.

1. $X_{\mathbf{j}} = \displaystyle\bigcup_{\mathbf{shape(i)} \subset \mathbf{shape(j)}} C_{\mathbf{i}}$.

2. The dimension of $X_{\mathbf{j}}$ is $|\mathbf{shape(j)}|$.

3. The Grassmannian $G(k,n) = X_{\{n-k+1,\ldots,n-1,n\}}$ is a Schubert variety!

Enumerative Geometry Revisited

Question. How many lines intersect four given lines in $\mathbb{R}^3$?

Translation. Given a line in $\mathbb{R}^3$, the family of lines intersecting it can be interpreted in $G(2,4)$ as the Schubert variety

$$X_{\{2,4\}} = \overline{\begin{pmatrix} * & 1 & 0 & 0 \\ * & 0 & * & 1 \end{pmatrix}}$$

with respect to a suitably chosen basis determined by the line.

Reformulated Question. How many subspaces $U \in G(2,4)$ are in the intersection of 4 copies of the Schubert variety $X_{\{2,4\}}$ each with respect to a different basis?

Modern Solution. Use Intersection Theory!

Intersection Theory

- Schubert varieties induce canonical elements of the cohomology ring for $G(k,n)$ called *Schubert classes*: $[X_{\mathbf{j}}]$.

- Multiplication of Schubert classes corresponds with intersecting Schubert varieties with respect to different bases.

$$[X_{\mathbf{i}}][X_{\mathbf{j}}] = \left[X_{\mathbf{i}}(B^1) \cap X_{\mathbf{j}}(B^2)\right]$$

- Schubert classes add and multiply just like *Schur functions*. Schur functions are a fascinating family of symmetric functions indexed by partitions which appear in many areas of math, physics, theoretical computer science.

- Expanding the product of two Schur functions into the basis of Schur functions can be done via linear algebra:

$$S_\lambda S_\mu = \sum c_{\lambda,\mu}^{\nu} S_\nu.$$

- The coefficients $c_{\lambda,\mu}^{\nu}$ are non-negative integers called the Littlewood-Richardson coefficients. In a 0-dimensional intersection, the coefficient of $[X_{\{1,2,\dots,k\}}]$ is the number of subspaces in $X_{\mathbf{i}}(B^1) \cap X_{\mathbf{j}}(B^2)$.

Enumerative Solution

Reformulated Question. How many subspaces $U \in G(2,4)$ are in the intersection of 4 copies of the Schubert variety $X_{\{2,4\}}$ each with respect to a different basis?

Enumerative Solution

Reformulated Question. How many subspaces $U \in G(2,4)$ are in the intersection of 4 copies of the Schubert variety $X_{\{2,4\}}$ each with respect to a different basis?

Solution.

$$\left[X_{\{2,4\}}\right] = S_{(1)} = x_1 + x_2 + \dots$$

By the recipe, compute

$$\left[X_{\{2,4\}}(B^1) \cap X_{\{2,4\}}(B^2) \cap X_{\{2,4\}}(B^3) \cap X_{\{2,4\}}(B^4)\right]$$

$$= S_{(1)}^4 = 2S_{(2,2)} + S_{(3,1)} + S_{(2,1,1)}.$$

Answer. The coefficient of $S_{2,2} = [X_{1,2}]$ is *2* representing the two lines meeting 4 given lines in general position.

Recap

1. $G(k, n)$ is the Grassmannian variety of k-dim subspaces in $\mathbb{R}^n$.

2. The Schubert varieties in $G(k, n)$ are nice projective varieties indexed by k-subsets of $[n]$ or equivalently by partitions in the $k \times (n - k)$ rectangle.

3. Geometrical information about a Schubert variety can be determined by the combinatorics of partitions.

4. Intersection theory applied to Schubert varieties can be used to solve problems in enumerative geometry.

Generalizations

"Nothing is more disagreeable to the hacker than duplication of effort. The first and most important mental habit that people develop when they learn how to write computer programs is to generalize, generalize, generalize.: –Neil Stephenson "In the Beginning was the Command Line"

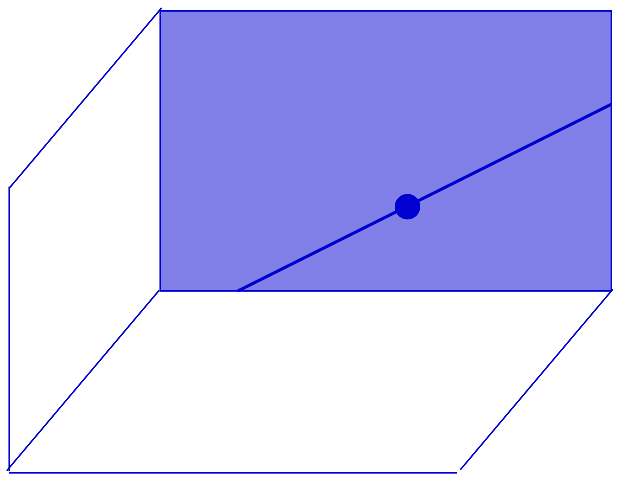
Same goes for mathematicians!

The Flag Manifold

Defn. A *complete flag* $F_\bullet = (F_1, \ldots, F_n)$ in $\mathbb{C}^n$ is a nested sequence of vector spaces such that $\dim(F_i) = i$ for $1 \le i \le n$. $F_\bullet$ is determined by an ordered basis $\langle f_1, f_2, \ldots f_n \rangle$ where $F_i = \mathrm{span}\langle f_1, \ldots, f_i \rangle$.

Example.

$$F_\bullet = \langle 6e_1 + 3e_2, \quad 4e_1 + 2e_3, \quad 9e_1 + e_3 + e_4, \quad e_2 \rangle$$



The Flag Manifold

Canonical Form.

$$F_\bullet = \langle 6e_1 + 3e_2,\ \ 4e_1 + 2e_3,\ \ 9e_1 + e_3 + e_4,\ \ e_2 \rangle$$

$$\approx \begin{bmatrix} 6 & 3 & 0 & 0 \\ 4 & 0 & 2 & 0 \\ 9 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 7 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\approx \langle 2e_1 + e_2,\ \ 2e_1 + e_3,\ \ 7e_1 + e_4,\ \ e_1 \rangle$$

$\mathcal{F}l_n(\mathbb{C}) :=$ *flag manifold* over $\mathbb{C}^n \subset \prod_{k=1}^n G(n,k)$

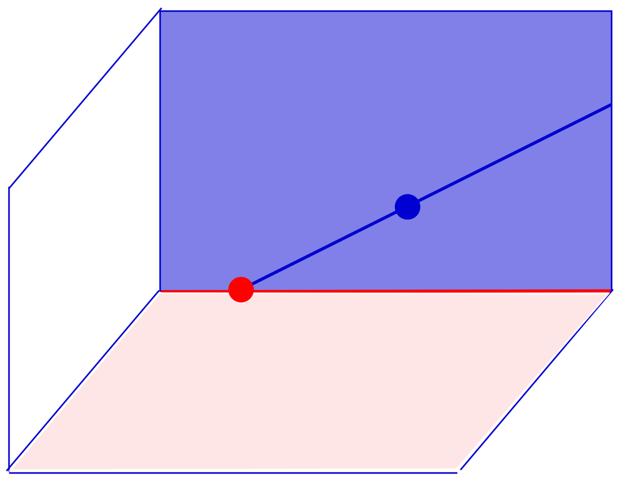
$= \{$complete flags $F_\bullet\}$

$= B \setminus GL_n(\mathbb{C}),\ \ B =$ lower triangular mats.

Flags and Permutations

Example. $F_\bullet = \langle 2e_1+e_2,\ \ 2e_1+e_3,\ \ 7e_1+e_4,\ \ e_1 \rangle \approx \begin{bmatrix} 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 7 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

Note. If a flag is written in canonical form, the positions of the leading 1's form a permutation matrix. There are 0's to the right and below each leading 1. This permutation determines the *position* of the flag $F_\bullet$ with respect to the reference flag $E_\bullet = \langle e_1,\ e_2,\ e_3,\ e_4 \rangle$.



Many ways to represent a permutation

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{bmatrix} = \mathbf{2341} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

matrix notation

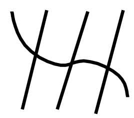
two-line notation

one-line notation

rank table

$$\begin{array}{cccc} * & \cdot & \cdot & \cdot \\ * & \cdot & \cdot & \cdot \\ * & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{array} =$$

1234



2341

$$= (\mathbf{1}, \mathbf{2}, \mathbf{3})$$

diagram of a permutation

string diagram

reduced word

The Schubert Cell $C_w(E_\bullet)$ in $\mathcal{F}l_n(\mathbb{C})$

Defn. $C_w(E_\bullet) =$ All flags $F_\bullet$ with $\mathbf{position}(E_\bullet, F_\bullet) = w$

$$= \{F_\bullet \in \mathcal{F}l_n \mid \dim(E_i \cap F_j) = \mathrm{rk}(w[i,j])\}$$

Example. $$F_\bullet = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 7 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \in C_{2341} = \left\{ \begin{bmatrix} * & 1 & 0 & 0 \\ * & 0 & 1 & 0 \\ * & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} : * \in \mathbb{C} \right\}$$

Easy Observations.

- $\mathtt{dim}_{\mathbb{C}}(C_w) = l(w) = \#$ inversions of w.
- $C_w = w \cdot B$ is a B-orbit using the right B action, e.g.

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} b_{1,1} & 0 & 0 & 0 \\ b_{2,1} & b_{2,2} & 0 & 0 \\ b_{3,1} & b_{3,2} & b_{3,3} & 0 \\ b_{4,1} & b_{4,2} & b_{4,3} & b_{4,4} \end{bmatrix} = \begin{bmatrix} b_{2,1} & b_{2,2} & 0 & 0 \\ b_{3,1} & b_{3,2} & b_{3,3} & 0 \\ b_{4,1} & b_{4,2} & b_{4,3} & b_{4,4} \\ b_{1,1} & 0 & 0 & 0 \end{bmatrix}$$

The Schubert Variety $X_w(E_\bullet)$ in $\mathcal{F}l_n(\mathbb{C})$

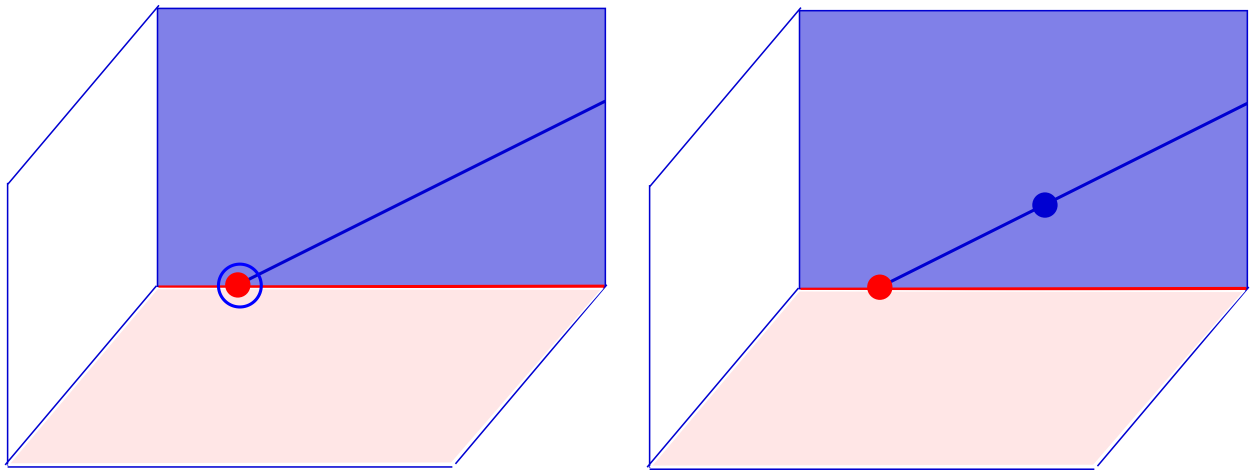
Defn. $X_w(E_\bullet)$ = Closure of $C_w(E_\bullet)$ under the Zariski topology

$$= \{F_\bullet \in \mathcal{F}l_n \mid \dim(E_i \cap F_j) \geq \mathrm{rk}(w[i,j])\}$$

where $E_\bullet = \langle e_1,\ e_2,\ e_3,\ e_4\ \rangle$.

Example. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & * & 1 & 0 \\ 0 & * & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \in X_{2341}(E_\bullet) = \overline{\left\{ \begin{bmatrix} * & 1 & 0 & 0 \\ * & 0 & 1 & 0 \\ * & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \right\}}$

Why?.



Five Fun Facts

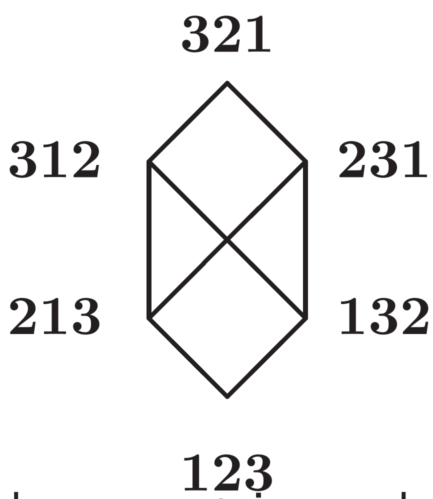
Fact 1. The closure relation on Schubert varieties defines a nice partial order.

$$X_w = \bigcup_{v \le w} C_v \quad = \bigcup_{v \le w} X_v$$

Bruhat order (Ehresmann 1934, Chevalley 1958) is the transitive closure of

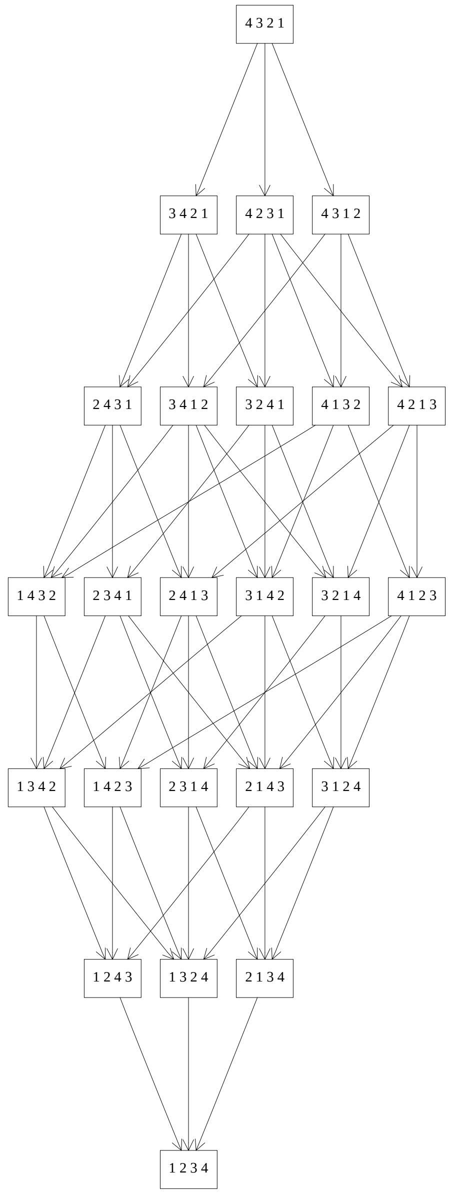
$$w < wt_{ij} \iff w(i) < w(j).$$

Example. Bruhat order on permutations in S_3.



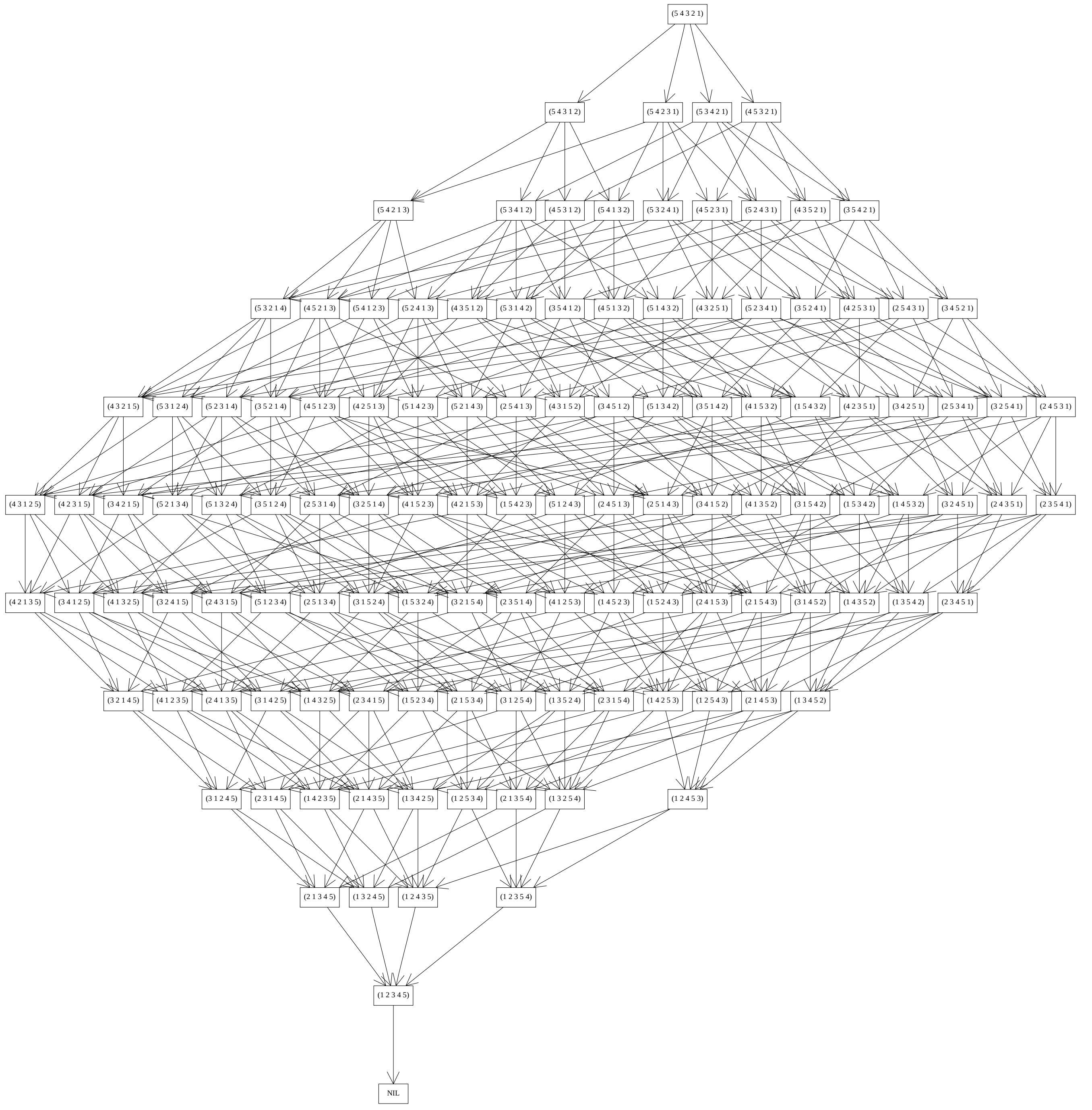
Observations. Self dual, rank symmetric, rank unimodal.

Bruhat order on S_4



Observations. Self dual, rank symmetric, rank unimodal.

Bruhat order on S_5



10 Fantastic Facts on Bruhat Order

1. Bruhat Order Characterizes Inclusions of Schubert Varieties
2. Contains Young's Lattice in S_∞
3. Nicest Possible Möbius Function
4. Beautiful Rank Generating Functions
5. $[x, y]$ Determines the Composition Series for Verma Modules
6. Symmetric Interval $[\hat{0}, w] \iff X(w)$ rationally smooth
7. Order Complex of (u, v) is shellable
8. Rank Symmetric, Rank Unimodal and k-Sperner
9. Efficient Methods for Comparison
10. Amenable to Pattern Avoidance

Singularities in Schubert Varieties

Defn. X_w is *singular* at a point $p \iff$
$\mathbf{dim} X_w = l(w) <$ dimension of the tangent space to X_w at p.

Observation 1. Every point on a Schubert cell C_v in X_w looks locally the same. Therefore, $p \in C_v$ is a singular point $\iff$ the permutation matrix v is a singular point of X_w.

Observation 2. The singular set of a varieties is a closed set in the Zariski topology. Therefore, if v is a singular point in X_w then every point in X_v is singular. The irreducible components of the singular locus of X_w is a union of Schubert varieties:

$$\mathbf{Sing}(X_w) = \bigcup_{v \in \mathbf{maxsing(w)}} X_v.$$

Singularities in Schubert Varieties

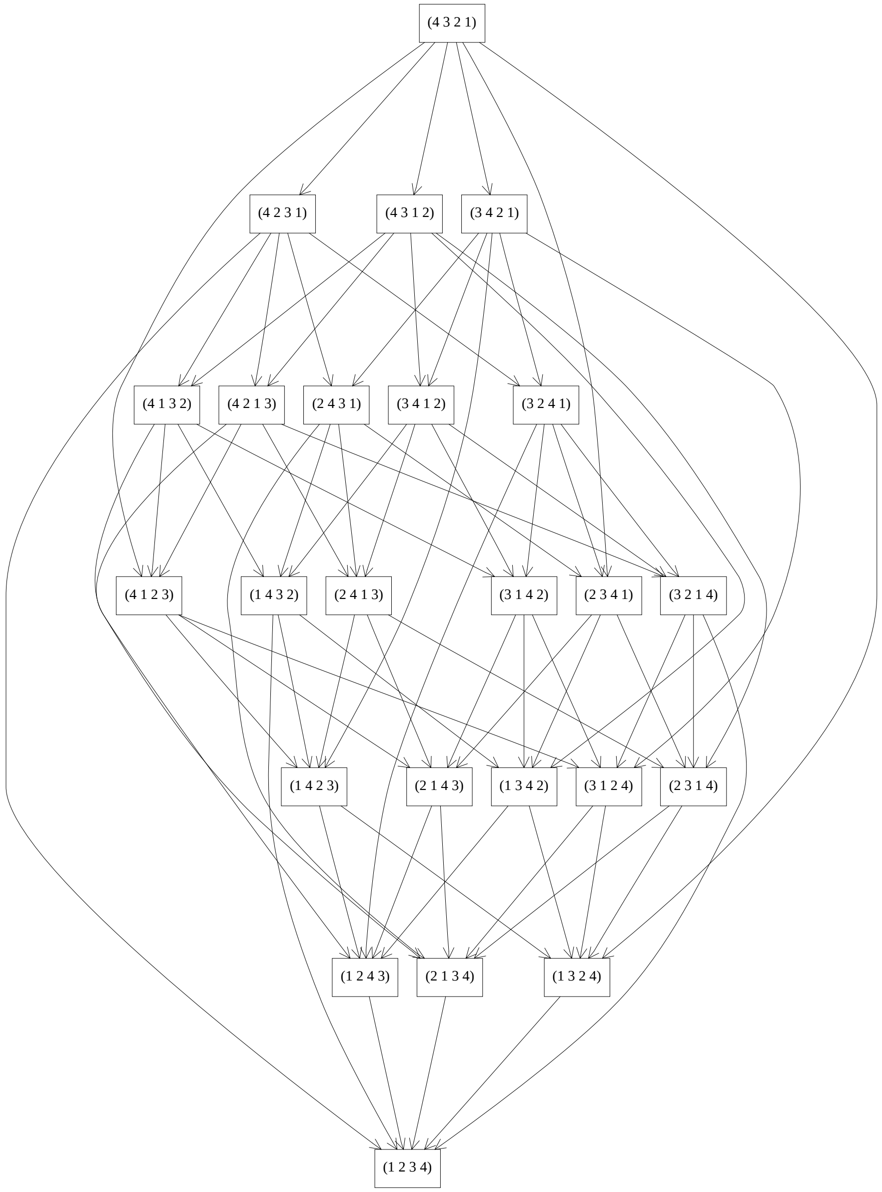
Fact 2. (Lakshmibai-Seshadri) A basis for the tangent space to X_w at v is indexed by the transpositions t_{ij} such that

$$vt_{ij} \leq w.$$

Definitions.

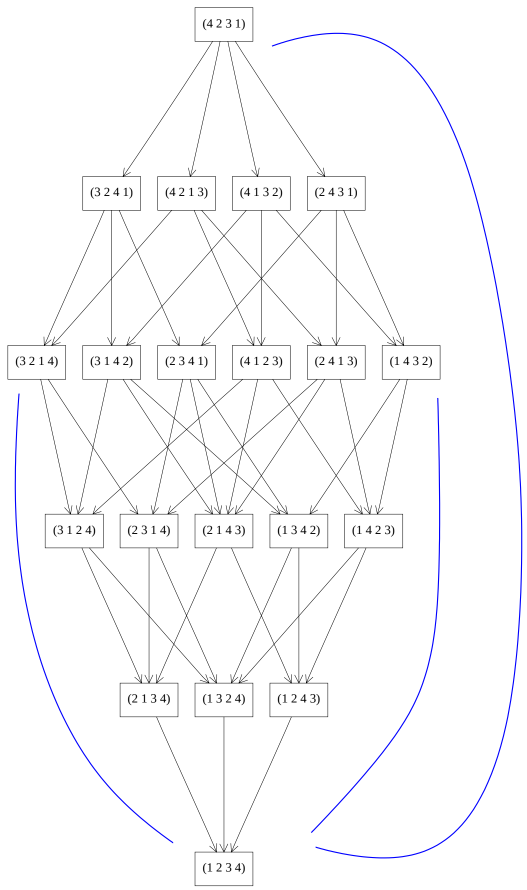
- Let T = invertible diagonal matrices. The T-fixed points in X_w are the permutation matrices indexed by $v \leq w$.
- If $v,\ vt_{ij}$ are permutations in X_w they are connected by a T-stable curve. The set of all T-stable curves in X_w are represented by the *Bruhat graph* on $[id, w]$.

Bruhat Graph in S_4



Tangent space of a Schubert Variety

Example. $T_{1234}(X_{4231}) = \text{span}\{x_{i,j} \mid t_{ij} \leq w\}$.



dim$X(4231)$=5 dim$T_{id}(4231) = 6 \implies X(4231)$ is singular!

Five Fun Facts

Fact 3. There exists a simple criterion for characterizing singular Schubert varieties using pattern avoidance.

Theorem: Lakshmibai-Sandhya 1990 (see also Haiman, Ryan, Wolper)
X_w is non-singular $\iff$ w has no subsequence with the same relative order as 3412 and 4231.

$w = 625431$ contains $6241 \sim 4231$ $\implies$ X_{625431} is singular

Example: $w = 612543$ avoids 4231 $\& 3412$ $\implies$ X_{612543} is non-singular

Five Fun Facts

Consequences of Fact 3.

- (Billey-Warrington, Kassel-Lascoux-Reutenauer, Manivel 2000) The bad patterns in w can also be used to efficiently find the singular locus of X_w.

- (Bona 1998, Haiman) Let v_n be the number of $w \in S_n$ for which $X(w)$ is non-singular. Then the generating function $V(t) = \sum_n v_n t^n$ is given by

$$V(t) = \frac{1 - 5t + 3t^2 + t^2\sqrt{1-4t}}{1 - 6t + 8t^2 - 4t^3}.$$

- (Billey-Postnikov 2001) Generalized pattern avoidance to all semisimple simply-connected Lie groups G and characterized smooth Schubert varieties X_w by avoiding these generalized patterns. Only requires checking patterns of types $A_3, B_2, B_3, C_2, C_3, D_4, G_2$.

Five Fun Facts

Fact 4. There exists a simple criterion for characterizing Gorenstein Schubert varieties using modified pattern avoidance.

Theorem: Woo-Yong (Sept. 2004)

X_w is Gorenstein $\iff$

- w avoids $\mathbf{31542}$ and $\mathbf{24153}$ *with Bruhat restrictions* $\{t_{15}, t_{23}\}$ and $\{t_{15}, t_{34}\}$

- for each descent d in w, the associated partition $\lambda_d(w)$ has all of its inner corners on the same antidiagonal.

Five Fun Facts

Fact 5. Schubert varieties are useful for studying the cohomology ring of the flag manifold.

Theorem (Borel): $H^*(\mathcal{F}l_n) \cong \dfrac{\mathbb{Z}[x_1, \ldots, x_n]}{\langle e_1, \ldots e_n \rangle}$.

- The symmetric function $e_i = \displaystyle\sum_{1 \le k_1 < \cdots < k_i \le n} x_{k_1} x_{k_2} \ldots x_{k_i}$.
- $\{[X_w] \mid w \in S_n\}$ form a basis for $H^*(\mathcal{F}l_n)$ over $\mathbb{Z}$.

Question. What is the product of two basis elements?

$$[X_u] \cdot [X_v] = \sum [X_w] c_{uv}^{w}.$$

Cup Product in $H^*(\mathcal{F}l_n)$

Answer. Use Schubert polynomials! Due to Lascoux-Schützenberger, Bernstein-Gelfand-Gelfand, Demazure.

- BGG: If $\mathfrak{S}_w \equiv [X_w] \mathrm{mod} \langle e_1, \ldots e_n \rangle$ then

$$\frac{\mathfrak{S}_w - s_i \mathfrak{S}_w}{x_i - x_{i+1}} \equiv [X_{ws_i}] \text{ if } l(w) < l(ws_i)$$

$$[X_{id}] \equiv x_1^{n-1} x_2^{n-2} \cdots x_{n-1} \equiv \prod_{i>j} (x_i - x_j) \equiv \ldots$$

  Here $\mathbf{deg}[X_w] = \mathbf{codim}(X_w)$.

- LS: Choosing $[X_{id}] \equiv x_1^{n-1} x_2^{n-2} \cdots x_{n-1}$ works best because product expansion can be done without regard to the ideal!

Schubert polynomials for S_4

$$
\begin{aligned}
\mathfrak{S}_{w_0(1234)} &= 1\\
\mathfrak{S}_{w_0(2134)} &= x_1\\
\mathfrak{S}_{w_0(1324)} &= x_2 + x_1\\
\mathfrak{S}_{w_0(3124)} &= x_1^2\\
\mathfrak{S}_{w_0(2314)} &= x_1x_2\\
\mathfrak{S}_{w_0(3214)} &= x_1^2x_2\\
\mathfrak{S}_{w_0(1243)} &= x_3 + x_2 + x_1\\
\mathfrak{S}_{w_0(2143)} &= x_1x_3 + x_1x_2 + x_1^2\\
\mathfrak{S}_{w_0(1423)} &= x_2^2 + x_1x_2 + x_1^2\\
\mathfrak{S}_{w_0(4123)} &= x_1^3\\
\mathfrak{S}_{w_0(2413)} &= x_1x_2^2 + x_1^2x_2\\
\mathfrak{S}_{w_0(4213)} &= x_1^3x_2\\
\mathfrak{S}_{w_0(1342)} &= x_2x_3 + x_1x_3 + x_1x_2\\
\mathfrak{S}_{w_0(3142)} &= x_1^2x_3 + x_1^2x_2\\
\mathfrak{S}_{w_0(1432)} &= x_2^2x_3 + x_1x_2x_3 + x_1^2x_3 + x_1x_2^2 + x_1^2x_2\\
\mathfrak{S}_{w_0(4132)} &= x_1^3x_3 + x_1^3x_2\\
\mathfrak{S}_{w_0(3412)} &= x_1^2x_2^2\\
\mathfrak{S}_{w_0(4312)} &= x_1^3x_2^2\\
\mathfrak{S}_{w_0(2341)} &= x_1x_2x_3\\
\mathfrak{S}_{w_0(3241)} &= x_1^2x_2x_3\\
\mathfrak{S}_{w_0(2431)} &= x_1x_2^2x_3 + x_1^2x_2x_3\\
\mathfrak{S}_{w_0(4231)} &= x_1^3x_2x_3\\
\mathfrak{S}_{w_0(3421)} &= x_1^2x_2^2x_3\\
\mathfrak{S}_{w_0(4321)} &= x_1^3x_2^2x_3
\end{aligned}
$$

Cup Product in $H^*(\mathcal{F}l_n)$

Key Feature. Schubert polynomials have distinct leading terms, therefore expanding any polynomial in the basis of Schubert polynomials can be done by linear algebra just like Schur functions.

Buch: Fastest approach to multiplying Schubert polynomials uses Lascoux and Schützenberger's transition equations. Works up to about $n = 15$.

Draw Back. Schubert polynomials don't prove c_{uv}^{w}'s are nonnegative (except in special cases).

Cup Product in $H^*(\mathcal{F}l_n)$

Another Answer.

- By intersection theory: $[X_u] \cdot [X_v] = [X_u(E_\bullet) \cap X_v(F_\bullet)]$

- Perfect pairing: $[X_u(E_\bullet)] \cdot [X_v(F_\bullet)] \cdot [X_{w_0 w}(G_\bullet)] = c_{uv}^w [X_{id}]$

$$||$$

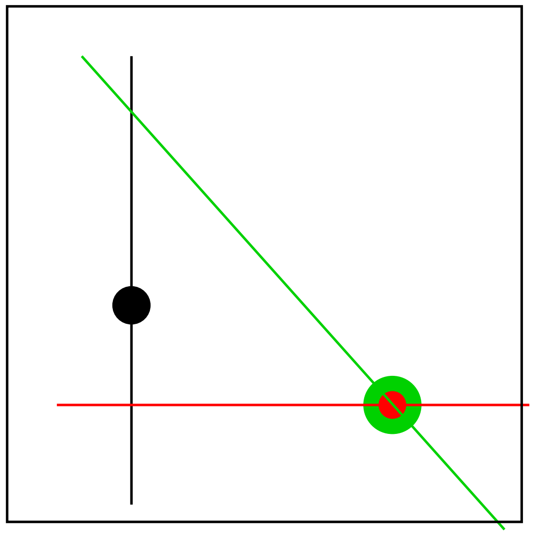
$$[X_u(E_\bullet) \cap X_v(F_\bullet) \cap X_{w_0 w}(G_\bullet)]$$

- The Schubert variety X_{id} is a single point in $\mathcal{F}l_n$.

Intersection Numbers: $c_{uv}^w = \# X_u(E_\bullet) \cap X_v(F_\bullet) \cap X_{w_0 w}(G_\bullet)$
Assuming all flags $E_\bullet$, $F_\bullet$, $G_\bullet$ are in sufficiently general position.

Intersecting Schubert Varieties

Example. Fix three flags $R_\bullet$, $G_\bullet$, and $B_\bullet$:

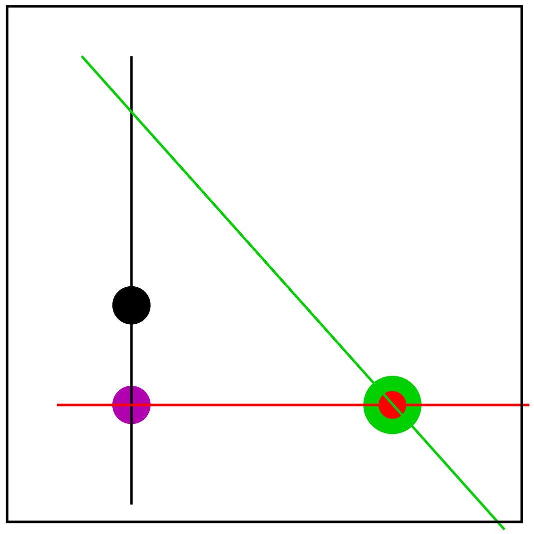


Find $X_u(R_\bullet) \cap X_v(G_\bullet) \cap X_w(B_\bullet)$ where u, v, w are the following permutations:

$$
\begin{array}{cccc} & R_1 & R_2 & R_3 \\ P_1 & & 1 & \\ P_2 & & & 1 \\ P_3 & 1 & & \end{array}
\qquad
\begin{array}{ccc} G_1 & G_2 & G_3 \\ & & 1 \\ & 1 & \\ 1 & & \end{array}
\qquad
\begin{array}{ccc} B_1 & B_2 & B_3 \\ & 1 & \\ 1 & & \\ & & 1 \end{array}
$$

Intersecting Schubert Varieties

Example. Fix three flags $R_\bullet$, $G_\bullet$, and $B_\bullet$:

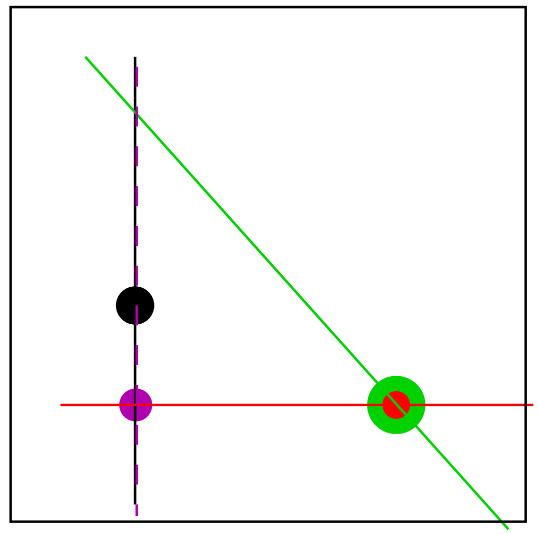


Find $X_u(R_\bullet) \cap X_v(G_\bullet) \cap X_w(B_\bullet)$ where u, v, w are the following permutations:

$$
\begin{array}{c|ccc|}
 & R_1 & R_2 & R_3 \\
\hline
P_1 & & 1 & \\
P_2 & & & 1 \\
P_3 & 1 & & \\
\hline
\end{array}
\qquad
\begin{array}{|ccc|}
G_1 & G_2 & G_3 \\
\hline
 & & 1 \\
 & 1 & \\
1 & & \\
\hline
\end{array}
\qquad
\begin{array}{|ccc|}
B_1 & B_2 & B_3 \\
\hline
 & 1 & \\
1 & & \\
 & & 1 \\
\hline
\end{array}
$$

Intersecting Schubert Varieties

Example. Fix three flags $R_\bullet$, $G_\bullet$, and $B_\bullet$:



Find $X_u(R_\bullet) \cap X_v(G_\bullet) \cap X_w(B_\bullet)$ where u, v, w are the following permutations:

$$
\begin{array}{c|ccc|}
 & R_1 & R_2 & R_3 \\ \hline
P_1 & & 1 & \\
P_2 & & & 1 \\
P_3 & 1 & & \\ \hline
\end{array}
\qquad
\begin{array}{|ccc|}
G_1 & G_2 & G_3 \\ \hline
 & & 1 \\
 & 1 & \\
1 & & \\ \hline
\end{array}
\qquad
\begin{array}{|ccc|}
B_1 & B_2 & B_3 \\ \hline
 & 1 & \\
1 & & \\
 & & 1 \\ \hline
\end{array}
$$

Answer. There is only one flag in $X_u(R_\bullet) \cap X_v(G_\bullet) \cap X_w(B_\bullet)$ so $c_{uv}^{w_0 w} = 1$.

Intersecting Schubert Varieties

Schubert's Problem. How many points are there usually in the intersection of d Schubert varieties if the intersection is 0-dimensional?

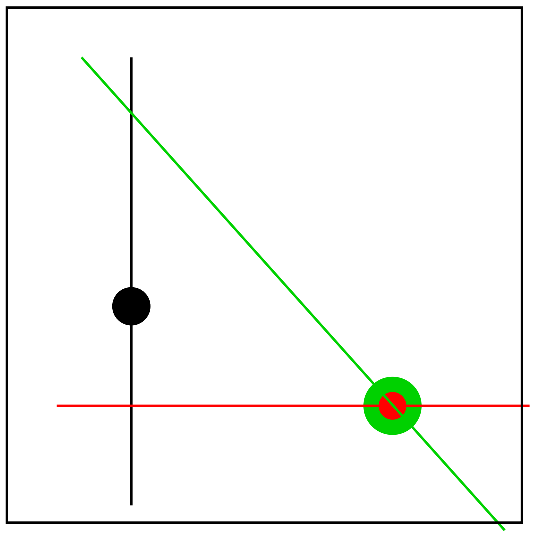
- Solving approx. n^d equations with $\binom{n}{2}$ variables is challenging!

Observation. We need more information on spans and intersections of flag components, e.g. $\dim(E^1_{x_1} \cap E^2_{x_2} \cap \cdots \cap E^d_{x_d})$.

Permutation Arrays

Theorem. (Eriksson-Linusson, 2000) For every set of d flags $E^1_\bullet, E^2_\bullet, \ldots, E^d_\bullet$, there exists a unique permutation array $P \subset [n]^d$ such that

$$\dim(E^1_{x_1} \cap E^2_{x_2} \cap \cdots \cap E^d_{x_d}) = \mathrm{rk}P[x].$$



	R_1	R_2	R_3
B_1			
B_2			
B_3	1	1	1

G_1

	R_1	R_2	R_3
B_1			
B_2			1
B_3	1	1	2

G_2

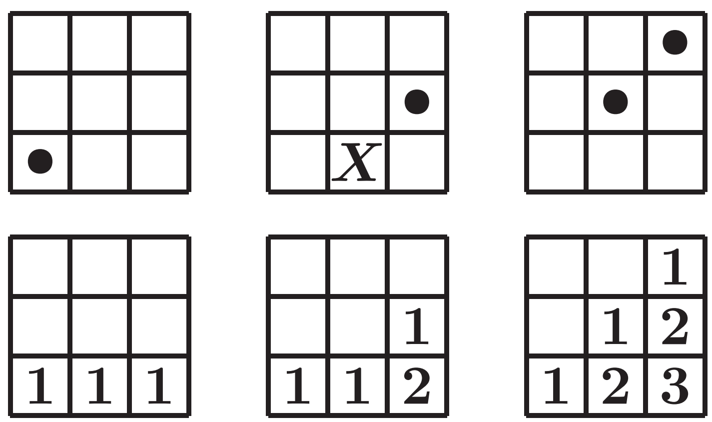
	R_1	R_2	R_3
B_1			1
B_2		1	2
B_3	1	2	3

G_3

Totally Rankable Arrays

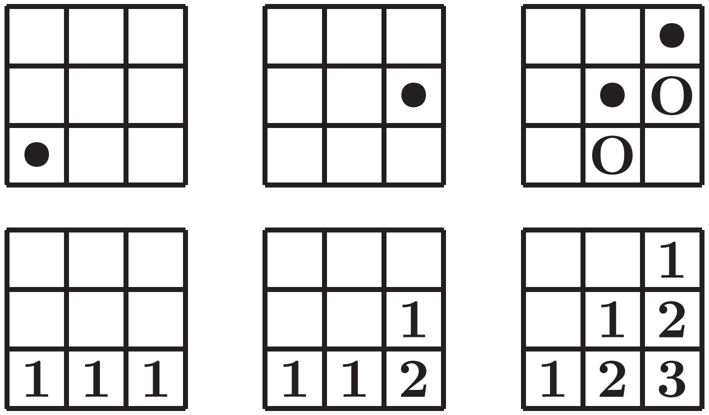
Defn. For $P \subset [n]^d$,

- $\mathrm{rk}_j P = \#\{k \mid \exists x \in P \text{ s.t. } x_j = k\}$.
- P is *rankable* of rank r if $\mathrm{rk}_j(P) = r$ for all $1 \le j \le d$.
- $y = (y_1, \ldots, y_d) \preceq x = (x_1, \ldots, x_d)$ if $y_i \le x_i$ for each i.
- $P[x] = \{y \in P \mid y \preceq x\}$
- P is *totally rankable* if $P[x]$ is rankable for all $x \in [n]^d$.



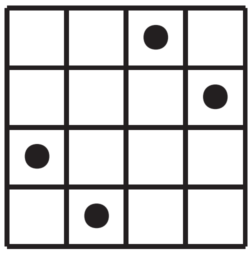
- Union of dots is totally rankable. Including X it is not.

Permutation Arrays



• Points labeled O are redundant, i.e. including them gives another totally rankable array with same rank table.

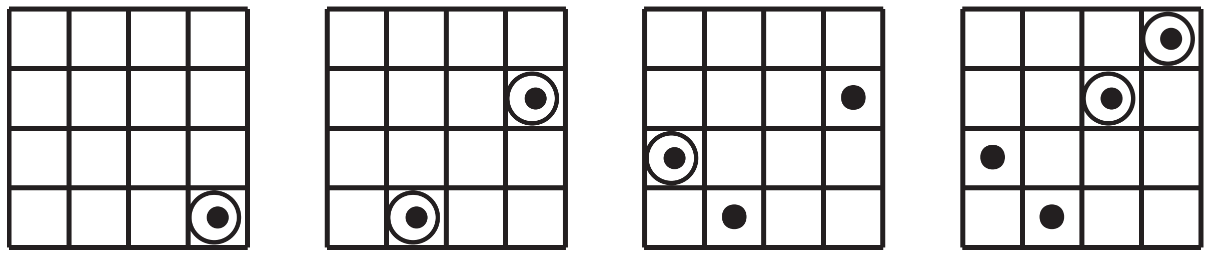
Defn. $P \subset [n]^d$ is a *permutation array* if it is totally rankable and has no redundant dots.

 $\in [4]^2.$

Open. Count the number of permutation arrays in $[n]^k$.

Permutation Arrays

Theorem. (Eriksson-Linusson) Every permutation array in $[n]^{d+1}$ can be obtained from a unique permutation array in $[n]^d$ by identifying a sequence of antichains.



This produces the 3-dimensional array

$$P = \{(4,4,1), (2,4,2), (4,2,2), (3,1,3), (1,4,4), (2,3,4)\}.$$

			4
		4	2
3			
	2		1

Unique Permutation Array Theorem

Theorem. (Billey-Vakil 2005) If

$$X = X_{w^1}(E^1_\bullet) \cap \cdots \cap X_{w^d}(E^d_\bullet)$$

is nonempty 0-dimensional intersection of d Schubert varieties with respect to flags $E^1_\bullet, E^2_\bullet, \ldots, E^d_\bullet$ in general position, then there exists a unique permutation array $P \in [n]^{d+1}$ such that

$$X = \{F_\bullet \mid \dim(E^1_{x_1} \cap E^2_{x_2} \cap \cdots \cap E^d_{x_d} \cap F_{x_{d+1}}) = \mathrm{rk}P[x].\} \qquad (1)$$

Furthermore, we can recursively solve a family of equations for X using P.

Open Problem. Can one find a finite set of rules for moving dots in a 3-d permutation array which determines the c^w_{uv}'s analogous to one of the many Littlewood-Richardson rules?

Recent Progress. Izzet Coskun's Mondrian tableaux.

Generalizations of Schubert Calculus for G/B

1997-2007: A Highly Productive Decade.

$$\left\{ \begin{array}{l} \text{A: } GL_n \\ \text{B: } SO_{2n+1} \\ \text{C: } SP_{2n} \\ \text{D: } SO_{2n} \\ \text{Semisimple Lie Groups} \\ \text{Kac-Moody Groups} \\ \text{GKM Spaces} \end{array} \right\} \times \left\{ \begin{array}{l} \text{cohomology} \\ \text{quantum} \\ \text{equivariant} \\ \text{K-theory} \\ \text{eq. K-theory} \end{array} \right\}$$

Contributions from: Bergeron, Billey, Brion, Buch, Carrell, Ciocan-Fontainine, Coskun, Duan, Fomin, Fulton, Gelfand, Goldin, Graham, Griffeth, Guillemin, Haibao, Haiman, Holm, Huber, Kirillov, Knutson, Kogan, Kostant, Kresh, S. Kumar, A. Kumar, Lam, Lascoux, Lenart, Miller, Peterson, Pitti, Postnikov, Ram, Robinson, Shimozono, Sottile, Sturmfels, Tamvakis, Vakil, Winkle, Yong, Zara...

See also A. Yong's slides on "Enumerative Formulas in Schubert Calculus"

http://math.berkeley.edu/ ayong/slides.html

Some Recommended Further Reading

1. "Schubert Calculus" by S. L. Kleiman; Dan Laksov The American Mathematical Monthly, Vol. 79, No. 10. (Dec., 1972), pp. 1061-1082.

2. "Young Tableaux" by William Fulton, London Math. Soc. Stud. Texts, Vol. 35, Cambridge Univ. Press, Cambridge, UK, 1997.

3. "The Symmetric Group" by Bruce Sagan, Wadsworth, Inc., 1991.

4. "Numerical Schubert calculus" by Birkett Huber, Frank Sottile, and Bernd Sturmfels, Journal of Symbolic Computation, 26, (1998) pp. 767-788.

5. "Determining the Lines Through Four Lines" by Michael Hohmeyer and Seth Teller, Journal of Graphics Tools, 4(3):11-22, 1999.

6. "Flag arrangements and triangulations of products of simplices" by Sara Billey and Federico Ardila, to appear in Adv. in Math.

7. Recent work of Frank Sottile and Anton Leykin here at IMA

   `http://www.math.tamu.edu/~sottile/pages/Flags/Data/F37/We7W2W21.91.html`