# Coxeter-Knuth Classes and a Signed Little Bijection

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# Outline

Transition Equations for Stanley symmetric functions

(signed) Little Bijection on Reduced Words

Kraśkiewicz insertion and Coxeter-Knuth moves on Reduced Words

Shifted Dual Equivalence Graphs

Open Problems

# Background

- $G$= Classical Complex Lie Group: $SL_n, SP_{2n}, SO_{2n+1}, SO_{2n}$
- $B$ = Borel subgroup = upper triangular matrices in $G$
- $W$ = Weyl groups $\subset B_n$ = signed permutations
- $S$ = simple reflections generating $W$
- $T = \{wsw^{-1} : s \in S, w \in W\} \subset$ signed transpositions

# Monk/Chevalley formula

**Fact.** The Schubert classes $\mathfrak{S}_w,\ w \in W$ form a basis for $H^*(G/B, \mathbb{Q})$ and they satisfy Monk/Chevalley's formula

$$\mathfrak{S}_w \cdot \mathfrak{S}_s = \sum_{t \in T : l(wt) = l(w)+1} (\mathfrak{S}_s, f_t) \mathfrak{S}_{wt},$$

$s$ =simple reflection, and $f_t$ is a linear polynomial negated by $t$.

**Defn.** *Transition Equations* are recurrences for Schubert classes derived from Monk/Chevalley's formula.

# Transition Equations

**Type $A$.** (Lascoux-Schützenberger,1984) For all $w \neq id$, let $(r < s)$ be the largest pair of positions inverted in $w$ in lexicographic order. If $w$ has only one descent, then $\mathfrak{S}_w$ is the Schur polynomial $s_{\lambda(w)}(x_1, ..., x_r)$. Otherwise,

$$\mathfrak{S}_w = x_r \mathfrak{S}_{wt_{rs}} + \sum \mathfrak{S}_{w'}$$

where the sum is over all $w'$ such that $l(w) = l(w')$ and $w' = wt_{rs}t_{ir}$ with $0 < i < r$. Call this set $T(w)$.

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**Example.** If $w = 7325614$, then $r = 5$, $s = 7$

$$\mathfrak{S}_w = x_4 \mathfrak{S}_{7325416} + \mathfrak{S}_{7425316} + \mathfrak{S}_{7345216}$$

So, $T(w) = \{7425316, 7345216\}$.

# Transition Equations

**Types $B, C, D$.** Schubert classes for increasing signed permutations are Schur P- or Q-functions (Pragacz,'91).

Transition equations are similar (Billey-Haiman'94 + Billey'95):

$$\mathfrak{S}_w = z_r \mathfrak{S}_{wt_{r,s}} + \sum_{i<r} \mathfrak{S}_{vt_{ir}} + \sum_{i \neq r} \mathfrak{S}_{vs_{ir}} + \chi \mathfrak{S}_{vs_{rr}},$$

where the sum is over all such $w' = wt_{rs}t$ such that $l(w) = l(w')$. Call this set $T(w)$ again.

$\chi$ depends only on the root system: $\chi_B = 2, \ \chi_C = 1, \ \chi_D = 0$.

# Schur P,Q-functions

**Defn.** A *Schur function* $s_\lambda(X) = \sum_T x^T$ is the generating function over all column strict tableaux $T$ of partition shape $\lambda$ and $x^T = x_1^{n_1} x_2^{n_2} \cdots$ where $n_i$ is the number of $i$'s in $T$.

**Defn.** A *Schur Q-function* $Q_\lambda(X) = \sum_T x^T$ is the generating function over all circle tableaux $T$ of shifted partition shape $\lambda$. A *circle* tableau has weakly increasing rows and columns on $1^\circ < 1 < 2^\circ < 2 < \ldots$ with no two $i^\circ$'s in a row and no two $i$'s in a column. The *Schur P-function* is $P_\lambda(X) = 2^{\ell(\lambda)} Q_\lambda$.

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**Example.** $\lambda = (3,1)$ and $X = \{x_1, x_2\}$

$$s_{(3,1)} = x_1^3 x_2 + x_1^2 x_2^2 + x_1 x_2^3.$$

$$Q_{(3,1)} = 4x_1^3 x_2 + 8x_1^2 x_2^2 + 4x_1 x_2^3$$

# Reduced words

**Notation.** $S_n$ is generated by adjacent transpositions $s_i = (i, i+1)$. For $w = [w_1, ..., w_n] \in S_n$, let $R(w)$ be the set of *reduced words* for $w$.

**Example.** $w = [4, 1, 3, 2] = s_3 s_2 s_1 s_3 = s_3 s_2 s_3 s_1 = s_2 s_3 s_2 s_1$ and

$$R(w) = \{3213, 3231, 2321\}.$$

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**Theorem.** (Stanley, 1984)

$$|R([n, n-1, \ldots, 1])| = \frac{\binom{n}{2}!}{3^{n-2} 5^{n-3} \cdots (2n-1)^1}.$$

# Stanley Symmetric Functions

**Notation.** Let $1 \times w = [1, (1 + w_1), ..., (1 + w_n)]$.

**Defn.** The *Stanley symmetric function* $F_w$ is equivalently given by

1. (L-S) $F_w(x_1, x_2, \ldots) = \lim_{n\to\infty} \mathfrak{S}_{(1^n \times w)}$.

2. (L-S) If $w$ has at most 1 descent, $F_w = s_{\lambda(w)}$. Otherwise,

$$F_w = \sum_{w' \in T(1\times w)} F_{w'}.$$

3. (St) If $a = (a_1, ..., a_p) \in R(w)$, let $I(a)$ be the set of weakly increasing positive integer sequences $(i_1, ..., i_p)$ such that $i_j < i_{j+1}$ whenever $a_j < a_{j+1}$. Then

$$F_w = \sum_{a\in R(w)} \sum_{(i_1,...,i_p)\in I(a)} x_{i_1} x_{i_2} \cdots x_{i_p}.$$

# Combinatorics from Schubert Calculus

**Cor.** If $w$ has at most 1 descent then $|R(w)| = f^{\lambda(w)}$ where $f^{\lambda}$ is the number of standard tableaux of shape $\lambda$. Otherwise,

$$|R(w)| = \sum_{w' \in T(1\times w)} |R(w')|.$$

**Cor.** $|R(w)| = \sum_{\lambda} a_{\lambda,w} f^{\lambda}$.

**Thm.**(Edelman-Greene, 1987) The coefficient $a_{\lambda,w}$ counts the number of distinct $P$ tableaux that arise when inserting all reduced words for $w$ via EG-insertion (variation on RSK). Furthermore, for each such $P$ and standard tableau $Q$ of the same shape, there exists a unique $a \in R(w)$ which inserts to $P$ and has recording tableau $Q$.

**Question.** Is there a bijection from $R(w)$ to $\cup_{w' \in T(w)} R(w')$ which preserves the descent set, Coxeter-Knuth classes and the $Q$ tableau?

# Little's Bijection

**Question.** Is there a bijection from $R(w)$ to $\cup_{w' \in T(w)} R(w')$ which preserves the descent set, Coxeter-Knuth classes and the $Q$ tableau?

**Answer.** Yes! It's called Little's bijection named for David Little (Little, 2003) + (Hamaker-Young, 2013).

**Thomas Lam's Conjecture.**(proved by Hamaker-Young, 2013) Every reduced word for any permutation with the same $Q$ tableau is connected via Little bumps. Every communication class under Little bumps contains a unique reduced word for a unique fixed point free Grassmannian permutation.

Thus, the Little bumps are analogous to jeu de taquin!

# Stanley Symmetric Functions

**Notation.** If $w = [w_1, ..., w_n]$ is a signed permutation, then $w$ is *increasing* if $w_1 < w_2 < \ldots < w_n$.

**Defn.** The type $C$ Stanley symmetric function $F_w$ for $w \in B_n$ are given by the following equivalent conditions:

1. (Billey-Haiman, 1994)(T.K.Lam, 1994)(Fomin-Kirilov, 1996) Let $R(w)$ be the set of reduced words for $w \in B_n$. If $a = (a_1, ..., a_p) \in R(w)$, let $I(a)$ be the set of weakly increasing positive integer sequences $(i_1, ..., i_p)$ such that $i_{j-1} = i_j = i_{j+1}$ only occurs if $a_j$ is not bigger than both $a_{j-1}$ and $a_{j+1}$. Then
$$F_w^C = \sum_{a \in R(w)} \sum_{i=(i_1,...,i_p) \in I(a)} 2^{|i|} x_{i_1} x_{i_2} \cdots x_{i_p}.$$
2. (Billey, 1996) If $w$ is increasing, $F_w = Q_{\mu(w)}$. Otherwise,
$$F_w^C = \sum_{w' \in T(w)} F_{w'}^C.$$

# Combinatorics from Schubert Calculus

**Cor.** If $w$ is not increasing, $|R(w)| = \sum_{w' \in T(w)} |R(w')|$

**Cor.** $|R(w)| = \sum_{\lambda} c_{\lambda,w} g^{\lambda}$ where $g^{\lambda}$ is the number of *shifted* standard tableaux of shape $\lambda$ and $c_{\lambda,w}$ is a nonnegative integer.

**Thm.**(Kraśkiewicz, 1995) The coefficient $b_{\lambda,w}$ counts the number of distinct $P$ tableaux that arise when inserting all reduced words for $w$ via Kraśkiewicz-insertion (variation on EG). Furthermore, for each such $P$ and standard tableau $Q$ of the same shape, there exists a unique $a \in R(w)$ which inserts to $P$ and has recording tableau $Q$. (See also Haiman evacuation procedure.)

# Combinatorics from Schubert Calculus

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**Question.** Is there a bijection from $R(w)$ to $\cup_{w' \in T(w)} R(w')$ which preserves the *peak* set and the $Q$ tableau for signed permutations?

# Kraśkiewicz Insertion

**Notation.** $ws_0 = [\bar{w}_1, w_2, \ldots, w_n]$

**Example.** $w = [\bar{3}, 1, 4, 2] = s_2 s_1 s_0 s_3 s_2$

$P(21032) =$

| | | | |
|---|---|---|---|
| 3 | 1 | 0 | 2 |
| | 2 | | |

$Q =$

| | | | |
|---|---|---|---|
| 1 | 2 | 3 | 4 |
| | 5 | | |

# Signed Little Bijection

**Answer.** Yes! We define *signed Little bumps* $B^-_{(i,j)}$ acting on $R(w)$ provided $wt_{i,j}$ covered by $w$ in Bruhat order.

**Thm.**(Billey-Hamaker-Roberts-Young, 2014)

- The bump $B^-_{(r,s)} : R(w) \longrightarrow \cup_{w' \in T(w)} R(w')$ is a bijection preserving peak sets and shifted recording tableaux under Kraśkiewicz insertion.
- Each $Q$ determines a communication class: every reduced word $a$ such that $Q(a) = Q$ is connected via signed Little bumps.
- Every communication class contains a unique reduced word for a unique increasing signed permutation.

Again, the signed Little bumps are analogous to jeu de taquin on shifted shapes!

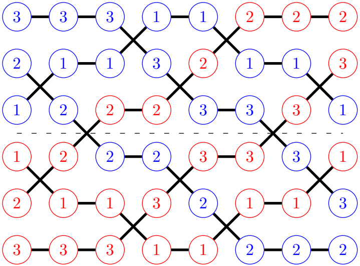
# The Algorithm

Given a reduced word, there is an associated reduced wiring diagram. If removing a crossing leaves another reduced wiring diagram, that crossing is a candidate to initiate a Little bump either pushing up or down. Pushing down (up) means reduce (increase) the corresponding letter in the word by 1.

- Check if the resulting word is reduced. If so, stop and return the new word.
- Otherwise, find the other point where the same two wires cross, and push them in the same direction.

**Signed Little Bijection.** Initiate a Little bump at the crossing $(r, s)$ corresponding to the lex largest inversion.

# The Algorithm in Pictures



# Coxeter-Knuth Relations

The proof that (signed) Little bumps preserve $Q$ tableaux uses a key lemma about how the $Q$ changes with elementary Coxeter relations.

**Coxeter relations.** (commutation relations) $ij \equiv ji$, $|i-j| > 1$, (braid relations) $0101 \equiv 1010$ and $i(i+1)i \equiv (i+1)i(i+1)$, $i > 0$.

**Defn.** *Coxeter-Knuth* moves on reduced words are minimally witnessed Coxeter relations that preserve $P$ tableau.

**Type A Coxeter-Knuth Moves.** (EG) Braids and witnessed commutations: $bac \equiv bca$, and $acb \equiv cab$ for $a < b < c$.

**Type B/C Coxeter-Knuth relations.** (K) 9 rules on windows of length 4 including $0101 \equiv 1010$ and $a(b+1)b(b+1) \equiv ab(b+1)b$ for $a < b$, but no other braids.

# Coxeter-Knuth Graphs and Dual Equivalence

**Defn.** The Coxeter-Knuth graph for $w$ has $V = R(w)$ and two reduced words are connected by an edge labeled $i$ if they agree in all positions except for a single Coxeter-Knuth relation starting in position $i$.

**Defn.** (Assaf, 2008) Dual equivalence graphs are graphs with labeled edges whose connected components are isomorphic to the graph on standard tableaux of a fixed partition shape with an edge labeled $i$ connecting any two vertices which differ by a transposition (i,i+1) or (i+1,i+2) with the third number on a diagonal in between the transposing pair.

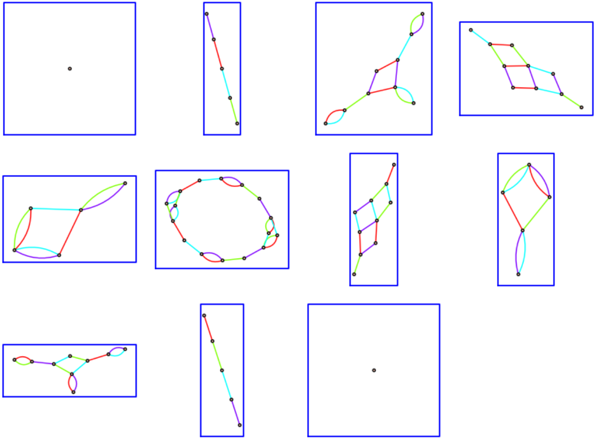
# Coxeter-Knuth Graphs and Dual Equivalence

**Thm.** The Coxeter-Knuth graphs in type $A$ are dual equivalence graphs and the isomorphism is given by the $Q$ tableaux in Edelman-Greene insertion.

In type $A$, this is a nice corollary of (Roberts, 2013) $+$ (Hamaker-Young, 2013).

**Thm.**(Chmutov, 2013) Stembridge's $A$-molecules are dual equivalence graphs and the edge labeling comes from labeling the Coxeter graph's edges consecutively.

# Coxeter-Knuth Graphs and Dual Equivalence Graphs



# Shifted Dual Equivalence

**Defn.** Given a permutation $\pi \in S_n$, define the *elementary shifted dual equivalence* $h_i$ for all $1 \leq i \leq n-3$ as follows. If $n = 4$, then $h_1(\pi)$ acts by swapping $x$ and $y$ in the cases below,

$$1x2y \quad x12y \quad 1x4y \quad x14y \quad 4x1y \quad x41y \quad 4x3y \quad x43y,$$

and $h_1(\pi) = \pi$ otherwise. If $n > 4$, then $h_i$ is the involution that fixes values not in $I = \{i, i+1, i+2, i+3\}$ and permutes the values in $I$ according to their relative order in $\pi$ as above.

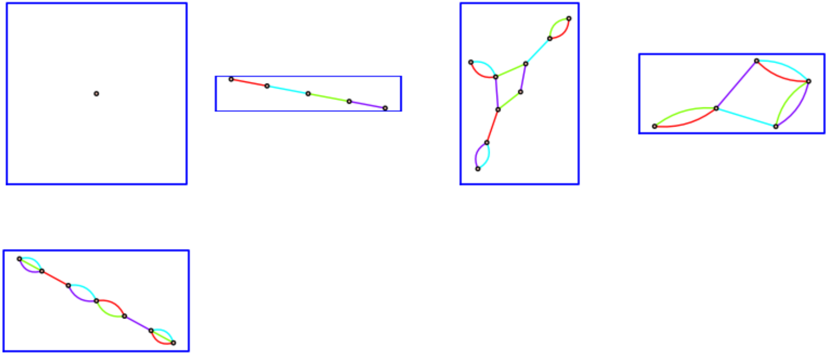
**Examples.** $h_1(24531) = 14532$, $h_2(25314) = 24315$, $h_3(314526) = 314526$.

# Shifted Dual Equivalence

**Thm.** (Haiman, 1992) The reading words of every standard shifted tableaux of shape $\lambda$ are connected via elementary shifted dual equivalence moves.

**Defn.** Define a *shifted dual equivalence graph* to be an edge colored graph which is isomorphic to the graph on standard shifted tableaux where the moves act on the reading word of the tableaux.

# Shifted Dual Equivalence Graphs for partitions of 7



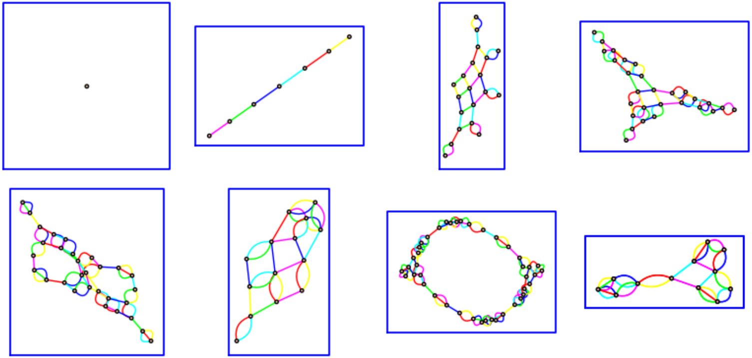
# Coxeter-Knuth Graphs and Shifted Dual Equivalence

**Thm.** The Coxeter-Knuth graphs in type $B$ are shifted dual equivalence graphs and the isomorphism is given by the $Q$ tableaux in Kraśkiewicz insertion.

**Thm.** Shifted dual equivalence graphs can be axiomitized using a local rule and a commutation rule:

- Every connected component using only 5 consecutive edge labels $h_i, h_{i+1}, h_{i+2}, h_{i+3}, h_{i+4}$ must be isomorphic to a connected component on standard shifted tableaux.
- The involutions $h_i$ and $h_j$ commute provided $|i - j| > 4$.

# Shifted Dual Equivalence Graphs for partitions of 9



# Open Problems

1. What does the (signed) Little bijection mean geometrically?
2. Is there a Little bump algorithm for all Weyl/Coxeter groups? See (Lam-Shimozono, 2005) for affine type $A$.
3. How can Coxeter-Knuth relations be defined independent of Lie type?
4. How can the Little bump algorithm be useful for Schubert calculus in analogy with jeu de taquin for Schur functions?
5. Are there other symmetric functions which expand nicely in the peak quasisymmetric functions which have a nice graph shifted dual equivalence graph structure?