Basic Schubert Calculus (Part 2)

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Combinatorial Algebraic Geometry
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Outline

Chow Cohomology Ring of $\mathcal{F}_n$

Monk's Fomula

Schubert Polynomials via Transition Equations

The Complete Solution to Hilbert's 15 Problem

Schubert Problems

Fix d permutations and d reference flags. The intersection of the corresponding Schubert varieties is a variety

$$Y = X_{w^1}(R^1_\bullet) \cap X_{w^2}(R^2_\bullet) \cap \cdots \cap X_{w^d}(R^d_\bullet).$$

Each of the irreducible components of Y will be rationally equivalent to translates of Schubert varieties.

Generalized Schubert Problem. How many Schubert varieties of each type appear as irreducible components of Y?

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Modern Solution. Use the theory of Chow Rings and Schubert polynomials to count the number of each type.

Historical Perspective

Thm.(Ehresmann, 1934) The complete flag manifold $\mathcal{F}_n \approx GL_n/B$ has a partition into Schubert cells C_w for $w \in S_n$ such that $X_w = \overline{C_w} = \bigcup_{v \leq w} C_v$ where $\leq$ is "Bruhat order". Also, the Poincaré polynomial of $H^*(GL_n/B, \mathbb{Z})$ is given by mapping $q \to q^2$ in $[n]_q! = \prod_{k=1}^{n}(1 + q + q^2 + \cdots + q^{k-1})$.

Thm.(Borel, 1953) $H^*(GL_n/B, \mathbb{Z})$ is isomorphic as a ring to the coinvariant algebra $\mathbb{Z}[x_1, \ldots, x_n]/\langle e_1, e_2, \ldots, e_n \rangle$ where e_k is the k^{th} elementary symmetric function.

Thm.(Chow, 1956) $H^*(GL_n/B, \mathbb{Z})$ has a basis given by classes related to Schubert varieties $\{[X_w] \mid w \in S_n\}$ and multiplication has an interpretation in terms of intersection of Schubert varieties.

Dream to Reality: Intersection theory

"Intersection Theory" (Fulton, 1998) completed the details on the isomorphism between the Chow ring and the cohomology ring.

Chow Ring of $\mathcal{F}_n$.

- Each variety $X \subset \mathcal{F}_n$ determines an equivalence class $[X]$, where $[X] = [Y]$ if X and Y are rationally equivalent varieties, e.g. $[X_w(B_\bullet)] = [X_w(R_\bullet)] = [X_w]$.
- Addition of classes is just formal addition.
- If $Z = Z_1 \cup Z_2 \cup \cdots \cup Z_d \subset \mathcal{F}_n$ is a variety with d irreducible components then $[Z] = [Z_1] + [Z_2] + \cdots + [Z_d]$.

Dream to Reality: Intersection theory

Chow Ring of $\mathcal{F}_n$.

- The classes $\{[X_w] : w \in S_n\}$ form a linear basis for a vector space over $\mathbb{Q}$ graded by codimension. (Chow's thm)

- Multiplication rule: For $R_\bullet$ and $B_\bullet$ in general position

$$[X_u] \cdot [X_v] = [X_u(R_\bullet) \cap X_v(B_\bullet)] = \sum c_{uv}^{w} [X_w].$$

The *structure constant* c_{uv}^{w} is the number of irreducible components of $X_u(R_\bullet) \cap X_v(B_\bullet)$ rationally equivalent to $X_w(G_\bullet)$.

Intersection Theory (Secret Sauce!)

Let $w_0 = n(n-1)\dots 321 \in S_n$.

- By intersecting sets:
  $[X_w] \cdot [X_{w_0 w}] = [X_w(E_\bullet) \cap X_{w_0 w}(F_\bullet)] = [\{w\}] = [X_{id}]$
- The Schubert variety X_{id} is a single point in $\mathcal{F}_n$.
- Perfect pairing: $[X_u(E_\bullet)] \cdot [X_v(F_\bullet)] \cdot [X_{w_0 w}(G_\bullet)] = c_{uv}^w [X_{id}]$

$$\|$$

$$[X_u(E_\bullet) \cap X_v(F_\bullet) \cap X_{w_0 w}(G_\bullet)]$$

Structure constants: $c_{uv}^w = \# X_u(E_\bullet) \cap X_v(F_\bullet) \cap X_{w_0 w}(G_\bullet) \in \mathbb{Z}_{\geq 0}$.
Assuming all flags $E_\bullet$, $F_\bullet$, $G_\bullet$ are in sufficiently general position.

Solving Schubert Problems

Summary. Chow ring structure constants are the answers to 0-dimensional Schubert problems, and they can be computed via the ring homomorphism with $\mathbb{Z}[x_1, \ldots, x_n]/\langle e_1, e_2, \ldots, e_n \rangle$.

Goal. Map each class $[X_w]$ to a polynomial $\mathbb{Z}[x_1, \ldots, x_n]$ of degree $\mathrm{codim}(X_w) = \binom{n}{2} - inv(w)$.

Notation. $[X_w] \longrightarrow \mathfrak{S}_{w_0 w}(x_1, \ldots, x_n)$ (*Schubert polynomial*)

Monk's Perspective (1959)

Monk's Introduction says:

> *The present work was suggested by a comparison of the results of Ehresmann and Borel. Its purpose is to study the geometrical properties of the flag manifold, mainly by the methods of classical algebraic geometry, and to exhibit the relation between the different forms of the basis obtained by these authors.* "

Thm.(Monk, 1959) For $v \in S_n$ and $s_r = (r \leftrightarrow r+1)$

$$\mathfrak{S}_v \cdot \mathfrak{S}_{s_r} = \sum_{i \le r < j : \mathrm{inv}(vt_{ij}) = \mathrm{inv}(v)+1} \mathfrak{S}_{vt_{ij}}.$$

All Schubert classes are determined by this formula along with the "natural choices": $\mathfrak{S}_{s_r} = -(x_1 + x_2 + \ldots + x_r)$, but signs and multiplicities occur.

Monk's Formula

Thm.(Monk, 1959) For $v \in S_n$ and $s_r = (r \leftrightarrow r+1)$

$$\mathfrak{S}_v \cdot \mathfrak{S}_{s_r} = \sum_{i \leq r < j : \mathrm{inv}(vt_{ij}) = \mathrm{inv}(v)+1} \mathfrak{S}_{vt_{ij}}.$$

Proof Sketch. $\mathfrak{S}_{s_r}$ corresponds with the codimension 1 Schubert variety indexed by $w_0 s_r = n \dots r(r+1)..321$. Intersections with $X_{w_0 s_r}$ can be computed by analyzing flags concretely.

Note, intersections with X_{w_0} are trivial since $X_{w_0} = \mathcal{F}_n \implies \mathfrak{S}_{id} = 1$.

A Recurrence Relation for Schubert Polynomials

Thm.(Monk, 1959) For $v \in S_n$ and $s_r = (r \leftrightarrow r+1)$

$$\mathfrak{S}_v \cdot \mathfrak{S}_{s_r} = \sum_{i \le r < j : \mathrm{inv}(vt_{ij}) = \mathrm{inv}(v)+1} \mathfrak{S}_{vt_{ij}}.$$

Alternative Choice.: $\mathfrak{S}_{s_r} = x_1 + x_2 + \ldots + x_r$ leads to a recurrence for Schubert polynomials with all nonnegative coefficients!

Transition Equation. Set $\mathfrak{S}_{id} = 1$. For all $w \neq id$, let $(r < s)$ be the lex largest pair such that $w(r) > w(s)$.Then

$$\mathfrak{S}_w = x_r \mathfrak{S}_{wt_{rs}} + \sum \mathfrak{S}_{w'}$$

where the sum is over all w' such that $\mathrm{inv}(w) = \mathrm{inv}(w')$ and $w' = wt_{rs}t_{ir}$ with $0 < i < r$. Call this set $T(w)$.

Classical Transition Equation

Thm.(Lascoux-Schützenberger,1984) Set $\mathfrak{S}_{id} = 1$. For all $w \neq id$, let $(r < s)$ be the lex largest pair such that $w(r) > w(s)$. Then

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Example. If $w = 7325614$, then $r = 5$, $s = 7$

$$\mathfrak{S}_w = x_5 \mathfrak{S}_{7325416} + \mathfrak{S}_{7425316} + \mathfrak{S}_{7345216}$$

So, $T(w) = \{7425316, 7345216\}$.

Classical Transition Equation

Example. If $w = 7325614$, then $r = 5$, $s = 7$

$$\mathfrak{S}_w = x_5 \mathfrak{S}_{7325416} + \mathfrak{S}_{7425316} + \mathfrak{S}_{7345216}$$

...

$$\begin{aligned} =& x_1^6 x_2^2 x_3 x_4^2 x_5^2 + x_1^6 x_2^2 x_3^2 x_4 x_5^2 + x_1^6 x_2^3 x_3 x_4 x_5^2 \\ & x_1^6 x_2^2 x_3^2 x_4^2 x_5 + x_1^6 x_2^3 x_3 x_4^2 x_5 + x_1^6 x_2^3 x_3^2 x_4 x_5. \end{aligned}$$

Observations. The reverse lex largest term $x_1^6 x_2^2 x_3 x_4^2 x_5^2$

corresponds with the exponent vector $(6, 2, 1, 2, 2, 0, 0)$.

$$C_{7325614}(R_\bullet) = \left\{ \begin{bmatrix} * & * & * & * & * & * & 1 \\ * & * & 1 & 0 & 0 & 0 & 0 \\ * & 1 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & * & 1 & 0 & 0 \\ * & 0 & 0 & * & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} : * \in \mathbb{C} \right\}$$

Schubert Polynomial Basis

Def. The *code of the permutation* $w \in S_n$ is

$$\mathrm{code(w)} = (c_1, c_2, \ldots, c_n)$$

where c_i is the number of inversions $(i < j)$ such that $w_i > w_j$ with i fixed.

The code of a permutation gives a bijection from S_n to *sub-staircase vectors* in

$$\{0, \ldots, n-1\} \times \{0, \ldots, n-2\} \times \cdots \{0\}.$$

Observation. Up to trailing fixed points, every monomial in

$\mathbb{Z}[x_1, x_2, \ldots]$ corresponds with a unique Schubert polynomial in S_∞ with that leading term. Therefore, Schubert polynomials are a basis for all polynomials in $\mathbb{Z}[x_1, x_2, \ldots]$.

Modern Schubert Calculus

Schubert Solutions. Given permutations $u, v \in S_\infty$, find the expansion

$$\mathfrak{S}_u \mathfrak{S}_v = \sum c^w_{w_0 u, w_0 v} \mathfrak{S}_w$$

Therefore, the structure constants c^w_{uv} can be found by linear algebra and the Transition Equation!

Next Frontier. What is the best way to find the c^w_{uv}?

Warning on terminology!

Next Frontier. What is the best way to find the c_{uv}^{w}?

Warning. I never say "It is an open problem to find a combinatorial interpretation for the c_{uv}^{w}'s". They already count something!

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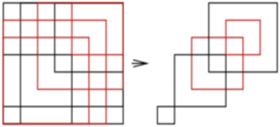
Warning. I never say “It is an open problem to find a combinatorial interpretation for the c_{uv}^{w}’s”. They already count something!

Plus, Izzet Coskun’s claimed his Mondrian tableaux fit that description. Can his algorithm be made more clear?

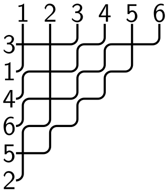


Pipe Dreams

History. Schubert polynomials were originally defined by Lascoux-Schützenberger early 1980's. Via work of Billey-Jockusch-Stanley, Fomin-Stanley, Fomin-Kirillov, Billey-Bergeron in the early 1990's we know the following equivalent definition.

Thm. For $w \in S_n$, $\mathfrak{S}_w(x_1, x_2, \ldots x_n) = \sum_{D \in RP(w)} x^D$ where $RP(w)$ are the *reduced pipe dreams* for w, aka *rc-graphs*.

Example. A reduced pipe dream D for $w = 314652$ where $x^D = x_1^3 x_2 x_3 x_5$.



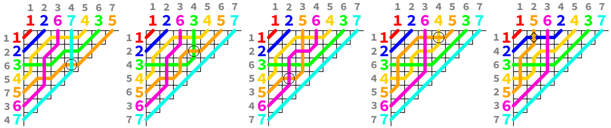
Pipe Dream Transition Equation

Bijective Proof. (Billey-Holroyd-Young, 2019) Based on David Little's bumping algorithm, the bounded bumping algorithm applied to the (r,s) crossing in a reduced pipe dream for w to bijectively prove

$$\mathfrak{S}_w = x_r \mathfrak{S}_{wt_{rs}} + \sum_{w' \in T(w)} \mathfrak{S}_{w'}.$$

Equivalently, we have a bijection

$$RP(w) \longrightarrow RP(wt_{rs}) \cup \bigcup_{w' \in T(w)} RP(w').$$

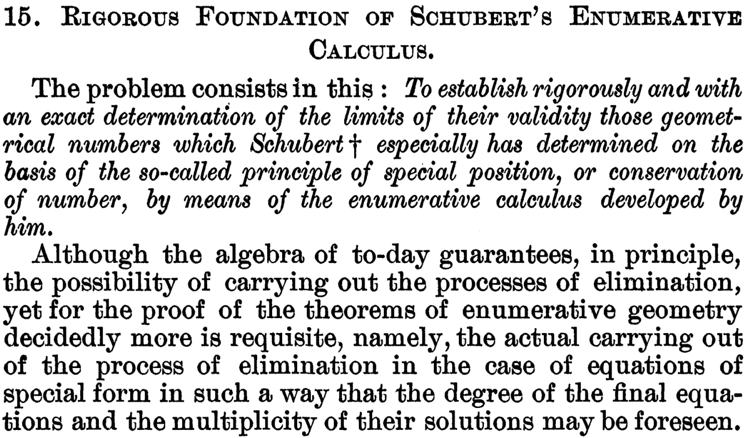


Note: This algorithm is not the same as the one used in Billey-Bergeron (1993) to bijectively prove Monk's formula!

Hilbert's 15th Problem

Mathematical Problems by Professor David Hilbert.

Lecture delivered at the ICM, 1900. (Bull.AMS)



15. RIGOROUS FOUNDATION OF SCHUBERT'S ENUMERATIVE CALCULUS.

The problem consists in this: *To establish rigorously and with an exact determination of the limits of their validity those geometrical numbers which Schubert* † *especially has determined on the basis of the so-called principle of special position, or conservation of number, by means of the enumerative calculus developed by him.*

Although the algebra of to-day guarantees, in principle, the possibility of carrying out the processes of elimination, yet for the proof of the theorems of enumerative geometry decidedly more is requisite, namely, the actual carrying out of the process of elimination in the case of equations of special form in such a way that the degree of the final equations and the multiplicity of their solutions may be foreseen.

Conclusion

Many Thanks!