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If $r = 1$, then $c_{ws_{24}} = 2$; if $r = 3$, then $c_{ws_{24}} = 1$; and if $r = 6$, then $c_{ws_{24}} = 0$.

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Some of the methods we use to prove Theorem 4 are similar to the recent work of Pragacz and Ratajski [19] where they obtain Pieri-type formulas for isotropic Grassmannians (see also [20]).

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One can verify (5.21) by noting $\partial_0^C \frac{-1}{2} p_1(Z)$ equals 1 and $\partial_i^B \frac{-1}{2} p_1(Z)$ equals 0 for all $i \geq 1$. Hence by the definition of Schubert polynomials $\mathfrak{B}_{\sigma_0}$ equals $\frac{-1}{2} p_1(Z)$.

The next corollary is a restatement of Chevalley's product formula in the special case the root system is type B, C or D. Note this form of the formula was also given by Hiller and Boe [9] for types B and C.

Corollary 11 (Analog of Monk's formula). *Let $\{S_w\}$ be a family of Schubert classes (or polynomials) of a fixed type B, C, or D and assume $S_{\sigma_r} = 2p_1(X) + z_1 + z_2 + \ldots + z_r$. Then we have*

(5.22)
$$S_{\sigma_r} S_w = \sum_{\substack{r<i<j \\ l(ws_{ij})=l(w)+1}} 2S_{ws_{ij}} + \sum_{\substack{r<i=j \\ l(ws_{ii})=l(w)+1}} \chi S_{ws_{ii}} + \sum_{\substack{i\leq r<j \\ l(ws_{ij})=l(w)+1}} S_{ws_{ij}} + \sum_{\substack{i\leq r<j \\ l(wt_{ij})=l(w)+1}} S_{wt_{ij}}$$

where the coefficient χ is given by (5.3)*. For the Schubert polynomials which do not have the usual form* (5.20) *have the following product formulas:*

$$\mathfrak{B}_{\sigma_0} \mathfrak{B}_w = \sum_{\substack{i\leq j \\ l(ws_{ij})=l(w)+1}} \mathfrak{B}_{ws_{ij}} \tag{5.23}$$

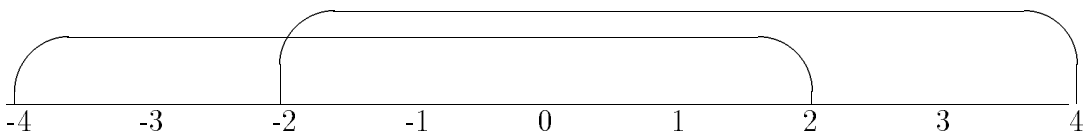
$$\mathfrak{D}_{\sigma_1} \mathfrak{D}_w = \sum_{\substack{1\leq i<j \\ l(ws_{ij})=l(w)+1}} \mathfrak{D}_{ws_{ij}} \tag{5.24}$$

$$\mathfrak{D}_{\sigma_1} \mathfrak{D}_w = \sum_{\substack{1<i<j \\ l(ws_{ij})=l(w)+1}} \mathfrak{D}_{ws_{ij}} + \sum_{\substack{1=i<j \\ l(wt_{1j})=l(w)+1}} \mathfrak{D}_{wt_{1j}} \tag{5.25}$$

For example,

$$\mathfrak{B}_{\sigma_2} \mathfrak{B}_{\bar{2}\bar{1}\bar{3}4} = \mathfrak{B}_{\bar{4}\bar{1}\bar{3}2} + 2\mathfrak{B}_{\bar{2}\bar{1}\bar{4}3} + \mathfrak{B}_{\bar{2}\bar{4}\bar{3}1} \tag{5.26}$$

There is an easy way to remember the coefficients in Corollary 11. Draw the integer number line from $-j$ to j and draw arcs between the points i with $-j$ and j with $-i$. Then the coefficient of $S^*_{ws_{ij}}$ is the number of arcs over the point r. For example, the coefficient of $S^*_{ws_{24}}$ in $S_w S_{\sigma_r}$ is computed with the aid of the following diagram:



where r is the last descent of w, s is the largest position such that $w_s < w_r$, $v = wt_{rs}$ and $F^*_{vt} = 0$ unless $l(vt) = l(w)$ in which case $F^*_{vt} = F_{vt}$.

This recursion for Stanley symmetric functions terminates when the signed permutation is strictly increasing, in which case $G_{w(\mu)} = P_\mu$, $F_{w(\mu)} = Q_\mu$, or $E_{w(\nu)} = P_\mu$ using the same notation as in (5.7). Therefore, each Stanley symmetric function expands as the positive sum of Schur P- or Q-functions respectively. The coefficients which appear in this expansion are related to the multiplicities given by the Haiman correspondence [2] (also known as the B- or D-Edelman-Greene for their analogy with the Edelman-Greene correspondence [6]). In type B and C these coefficients are also given by Kraskiewicz insertion [11], and in type D these coefficients are also given by Lam insertion [12]. Therefore, this recursion for Stanley symmetric functions gives an efficient new way of computing the multiplicities which arise from the Haiman correspondence, Kraskiewicz insertion, or Lam insertion.

The following corollary has appeared in several places beginning with Chevalley's proof [4], see also [1, 3.17], [5, Corollaire 2].

Corollary 10 (Chevalley's Product Formula). *Let $\{S_w\}$ be a family of Schubert classes (or polynomials) for any root system. We have*

$$S_{\sigma_r} S_w = \sum_{\substack{\beta \in R_+ \\ l(w\sigma_\beta) = l(w)+1}} -(S_{\sigma_r}, \check{\beta}) S_{w\sigma_\beta}, \tag{5.18}$$

Proof. The result follows directly from Lemma 7. □

Note that S_{σ_r} is a well defined linear polynomial in the cohomology ring of G/B since all of the invariants for the Weyl groups of types B and D have degree at least 2. From the recursion in Theorem 4 one can see that almost all of the Schubert classes have the form

$$S_{\sigma_r} = 2p_1(X) + z_1 + \ldots + z_r. \tag{5.19}$$

The exceptions to this rule are

$$\begin{aligned} B_{\sigma_0} &= p_1(X) \\ D_{\sigma_{\widehat{1}}} &= p_1(X) \\ D_{\sigma_1} &= p_1(X) + z_1 \end{aligned} \tag{5.20}$$

In order to expand any of the products $S_{\sigma_r} S_w$ we need to write $p_1(X)$ in terms of the z variables. It was shown in [2] that the relationship between these two sets of variables is given by

$$p_1(X) = \frac{-1}{2} p_1(Z). \tag{5.21}$$

To show the recursion will terminate, we give an ordering on the elements of W_∞ such that w is the largest element appearing in (5.2) and there are only a finite number of terms less than w in the ordering which occur in the expansion. Given $u \in W_\infty$, let $LD(u)$ be the position of the last descent, and $u_{LD(u)}$ is then the value of u in position $LD(u)$. Define a partial ordering on the elements of W_∞ by $u <_{LD} w$ if $LD(u) < LD(w)$ or if $LD(u) = LD(w)$ and $u_{LD(u)} < w_{LD(w)}$.

We claim each Weyl group element appearing on the right hand side of (5.2) is less than w under this partial order. First, $v <_{LD} w$ since $v = wt_{rs}$ where s was chosen so that $w_s < w_r$ and no new descents are created. The elements vt_{ir} for $i < r$ will never have a descent past position r, and again $(vt_{ir})_r < w_r$ whenever $l(vt_{ir}) = l(w)$. Hence $vt_{ir} <_{LD} w$. By Lemma 3, the descent set of each vs_{ir} appearing on the right hand side of (5.2) will be contained in the descent set of v and $(vs_{ir})_r < w_r$. Therefore, $vs_{ir} <_{LD} w$ in the partial order.

Assume w is a signed permutation on n elements, then the full expansion of S_w using (5.2) only depends on elements in W_{n+r}. Clearly, v and vt_{ir} for $i < r$ in (5.2) are in W_n. By Lemma 2, we see $l(vs_{ir}) \neq l(v) + 1$, if $i > n + 1$, so either $vs_{ir} \in W_n$ or $i = n + 1$ and $vs_{ir} \in W_{n+1}$. Note that the signed permutation $vs_{(n+1)r}$ must have $\overline{(n+1)}$ as its r^{th} entry, hence its last descent is in a position strictly less than r. Therefore, recursively applying (5.2) gives terms which depend only on elements in W_{n+r}, which is a finite group. □

Definition. For each $w \in B_\infty$ and $v \in D_\infty$ respectively, the Stanley symmetric functions F_w and E_v are defined by

$$\begin{aligned} G_w(X) &= \mathfrak{B}_w(Z, X)|_{Z=0} \\ F_w(X) &= \mathfrak{C}_w(Z, X)|_{Z=0} \\ E_v(X) &= \mathfrak{D}_v(Z, X)|_{Z=0}, \end{aligned} \tag{5.16}$$

where $G(Z, X)|_{Z=0}$ means set all the variables $z_i = 0$.

The Stanley symmetric functions were first defined in [7] and have also been studied in [2, 8, 12, 13, 22]. Corollary 9 follows directly from the definition.

Corollary 9. *The Stanley symmetric functions have the following recursive formulas:*

$$\begin{aligned} G_w &= 2G^*_{vs_{rr}} + \sum_{i<r} G^*_{vt_{ir}} + \sum_{i\neq r} G^*_{vs_{ir}} \\ F_w &= F^*_{vs_{rr}} + \sum_{i<r} F^*_{vt_{ir}} + \sum_{i\neq r} F^*_{vs_{ir}} \\ E_w &= \sum_{i<r} E^*_{vt_{ir}} + \sum_{i\neq r} E^*_{vs_{ir}} \end{aligned} \tag{5.17}$$

To do this, set $y = \sigma_{a_p} \cdots \sigma_{a_{r+1}}$, then $v = wy\sigma_{a_r}y^{-1}$. For any orthogonal transformation y, it is well known that $y\sigma_k y^{-1} = \sigma_{y(\alpha_k)}$, [10, 1.2]. Set $\beta = y(\alpha_{a_r})$, then $v = w\sigma_\beta$. Now we just compute,

$$\begin{aligned} \partial_k(z_i) &= \frac{z_i - \sigma_k z_i}{-\alpha_k} \\ &= \frac{z_i - \left(z_i - \frac{2(z_i,\alpha_k)\alpha_k}{(\alpha_k,\alpha_k)}\right)}{-\alpha_k} \\ &= -(z_i, \check{\alpha}_k). \end{aligned} \tag{5.13}$$

Then, using (5.13) and the fact that the inner product is invariant under the action of the Weyl group we have

$$\begin{aligned} \partial_{a_r} y^{-1}(f) &= \partial_{a_r}(c_{y(1)}z_1 + c_{y(2)}z_2 + \dots) \\ &= \sum -c_{y(i)}(z_i, \check{\alpha}_{a_r}) \\ &= \sum -c_{y(i)}(y(z_i), y(\check{\alpha}_{a_r})) \\ &= \sum -c_{y(i)}(z_{y(i)}, \check{\beta}) \\ &= -(f, \check{\beta}). \end{aligned} \tag{5.14}$$

□

Lemma 8. *For each family of Schubert polynomials (B, C, and D), we have the following formulas:*

$$z_k S_w = \sum_{i \neq k} \gamma_k(t_{ik}) S^*_{wt_{ik}} + \sum_{i \neq k} -S^*_{ws_{ik}} - \chi S^*_{ws_{kk}}. \tag{5.15}$$

where $\gamma_k(i)$ is -1 if $i < k$ or 1 if $k < i$, the constant χ is given by 5.3*, and again $S^*_{wt} = S_{wt}$ if $l(wt) = l(w) + 1$ and is equal to 0 otherwise.*

Proof. Using the notation for reflections, t_{ij} and s_{ij}, from Section 3 one can apply Lemma 7 to the linear polynomial $f = z_k$. Then one can easily check $(z_k, e_j - e_i) = 0$ unless $k = i$ or $k = j$ in which case it is given by γ_k. Also, $(z_k, e_i + e_j)$ is 1 if $k = i$ or $k = j$ and 0 otherwise. Recall, s_{ii} is not an element of the Weyl group of type D, thus the constant χ is 0 for type D. For type B, the reflection s_{ii} corresponds with the positive root e_i, hence the coefficient $(z_k, \check{e}_i)$ equals $2\delta_{ik}$. Similarly for type C, s_{ii} is the reflection over $2e_i$ so $(z_k, 2\check{e}_i) = \delta_{ik}$. □

Proof of Theorem 4. To obtain the formula in Theorem 4, apply Lemma 8 to $z_r S_v$ where r is the position of the last descent of w and $v = wt_{rs}$ as above. In this case, the only transposition t_{ir} such that $i > r$ and $l(vt_{ir}) = l(v) + 1$ according to Lemma 1 is for $i = s$ and then $vt_{rs} = w$.

Lemma 6. *Let f be any linear function in $\mathbb{Q}[z_1, z_2, \dots]$. Given any two elements $v, w \in W$ such that $l(v) = l(w) + 1$, let $a_1 \cdots a_p$ be a reduced word for v. We have*

$$\partial_v(fS_w) = \begin{cases} \partial_{a_r}\sigma_{a_{r+1}} \cdots \sigma_{a_p} f & \text{if } a_1 \cdots \widehat{a_r} \cdots a_p \in R(w) \text{ some } r \\ 0 & \text{otherwise.} \end{cases} \tag{5.8}$$

Proof. Applying a single divided difference operator to a product we have

$$\partial_i(fS_w) = (\sigma_i f)\partial_i S_w + S_w \partial_i f. \tag{5.9}$$

Applying any two divided difference operators to a linear function yields zero. Therefore, since f is linear, by successive application of (5.9) we have

$$\partial_v(fS_w) = \partial_{a_1} \dots \partial_{a_p}(fS_w) \tag{5.10}$$

$$= \left(v(f)\partial_v(S_w) + \sum_{r=1}^{p} \left(\sigma_{a_1} \cdots \partial_{a_r} \cdots \sigma_{a_p} f\right) \partial_{a_1} \cdots \widehat{\partial_{a_r}} \cdots \partial_{a_p} S_w \right). \tag{5.11}$$

We have assumed that S_w is homogeneous of degree equal to $l(w)$. Hence $\partial_v(S_w)$ is 0 since $l(v) > l(w)$. Furthermore, from (4.8) we have $\partial_v(S_w) = 1$ if $v = w$ and 0 if $v \neq w$ when $l(v) = l(w)$. This implies $\partial_{a_1} \cdots \widehat{\partial_{a_r}} \cdots \partial_{a_p} S_w$ equals 0 unless $a_1 \cdots \widehat{a_r} \cdots a_p \in R(w)$ in which case we get 1. One can check that there is at most one non-zero term in the sum (5.11) since $l(v) = l(w)+1$. Therefore, (5.11) reduces to (5.8). □

Lemma 7. *Let $f = \sum c_i z_i \in \mathbb{Q}[z_1, z_2, \dots]$, then*

$$fS_w = \sum_{\substack{\beta \in R_+ \\ l(w\sigma_\beta) = l(w)+1}} -(f, \check{\beta}) S_{w\sigma_\beta}, \tag{5.12}$$

where $\check{\beta} = 2\beta/(\beta, \beta)$ and $(f, \check{\beta})$ is the inner product of $\check{\beta}$ with the vector of coefficients $(c_1, c_2, \dots)$ of f.

Proof. By assumption (4.8),

$$fS_w = \sum_{l(v) = l(w)+1} \partial_v(fS_w) S_v.$$

For any v such that $l(v) = l(w) + 1$, let $a_1 a_2 \cdots a_p$ be a reduced word for v. By Lemma 6, we only need to compute $\partial_{a_r}\sigma_{a_{r+1}} \cdots \sigma_{a_p}(f)$ if $a_1 \cdots \widehat{a_r} \cdots a_p$ is a reduced word for w.

way to a polynomial of the form

$$S_w \equiv \sum c_{\alpha,\lambda} z^\alpha S_{w(\lambda)} \tag{5.5}$$

with non-negative integer coefficients $c_{\alpha,\lambda}$. Then using Pragacz's theorem take $S_{w(\lambda)} = Q_\lambda(X)$ (for example in type C) where X is another alphabet distinct from Z,

$$S_w \equiv \sum c_{\alpha,\lambda} z^\alpha Q_\lambda(X). \tag{5.6}$$

One could then multiply representatives in the ring

$$\mathbb{Z}[z_1, z_2, \dots ; p_1(X), p_3(X), \dots]$$

.

One should note that it is not true that $S_{w(\lambda)} = Q_\lambda(Z)$ where Z is the alphabet used in (4.6). In [2] the relationship between the Z variables and the X variables is made precise. However, for our purposes here we will assume they are distinct alphabets. It would be interesting to give a geometrical explanation for the relationship between the z's and the x's.

Theorem 5. *The Schubert polynomials found in* [2] *are exactly the representatives for the Schubert classes which can be derived from the transition equation.*

Proof. By induction, it is clear that any family of polynomials in $\mathbb{Z}[z_1, z_2, \dots ; p_1(X), p_3(X), \dots]$ which satisfy the transition equation and agree on the Grassmannian elements must be the same polynomials. By Theorems 3 and 4 of [2],

$$\begin{aligned} \mathfrak{B}_{w(\mu)} &= P_\mu(X) \\ \mathfrak{C}_{w(\mu)} &= Q_\mu(X) \\ \mathfrak{D}_{w(\nu)} &= P_\mu(X) \end{aligned} \tag{5.7}$$

in the notation used above.

The proof of the transition equations depends only on the three properties given in Proposition 2. In [2] it was shown that each family of Schubert polynomials of types B, C, and D form a $\mathbb{Z}$-basis for the ring $\mathbb{Z}[z_1, z_2, \dots ; p_1(X), p_3(X), \dots]$ where $p_k(X) = x_1^k + x_2^k \dots$ is a power sum, hence they satisfy Property 1. Properties 2 and 3 follow directly from the definition of Schubert polynomials given at the end of Section 4. Hence, the Schubert polynomials must also satisfy the transition equations. □

We now begin the proof of Theorem 4. We start by proving three lemmas. The outline of the proof follows the proof of Monk's formula in [16].

Theorem 4. *The Schubert classes S_w of types B, C, and D satisfy the following recursion formula:*

$$S_w = z_r S_v + \sum_{i<r} S^*_{vt_{ir}} + \sum_{i\neq r} S^*_{vs_{ir}} + \chi S^*_{vs_{rr}}, \tag{5.2}$$

where r is the last descent of w, s is the largest index such that $w_s < w_r$, $v = wt_{rs}$, χ depends only on the root system

$$\chi = \begin{cases} 2 & B \\ 1 & C \\ 0 & D, \end{cases} \tag{5.3}$$

and each $S^*_{vt} = 0$ unless $l(vt) = l(v) + 1 = l(w)$ for $v, t \in W_\infty$ in which case $S^*_{vt} = S_{vt}$. The recursion terminates in a finite number of terms consisting of a monomial in the z-variables times a Schubert class indexed by a strictly increasing signed permutation.

Pragacz has shown that Schur P- and Q-functions are representatives for the Schubert classes for the signed permutations which are strictly increasing. The Schur P- and Q- functions are symmetric functions in the ring generated by the odd power sums indexed by strictly decreasing partitions. Both $\{Q_\mu\}$ and $\{P_\mu\}$ form bases for $\mathbb{Q}[p_1, p_3, \ldots]$. For more information on these symmetric functions see [21, 23].

Let $w(\mu)$ be the signed permutation $\overline{\mu_1}\ \overline{\mu_2} \ldots \overline{\mu_l} 12 \ldots \widehat{\mu_1} \ldots \widehat{\mu_2} \ldots$, which has no descents. These special classes indexed by strictly increasing signed permutations will be called *Grassmannian elements* because they correspond to the Schubert varieties of the isotropic Grassmannian manifolds, G/P for P a maximal parabolic subgroup of G (associated to the first root in the basis). Let X be an infinite alphabet. Pragacz [18] shows there exists a ring homomorphism

$$\mathbb{Q}[p_1(X), p_3(X), p_5(X), \ldots] \to H^*(G/P, \mathbb{Q}) \tag{5.4}$$

such that the Schubert classes for Grassmannian elements are the images of Schur P- and Q-functions. In particular, Pragacz has shown that for type B (type C) the Schubert class indexed by $w(\mu)$ can be represented by $P_\mu(X)$ (Q_μ respectively. Also, if $\mu = (\mu_1 > \mu_2 > \cdots > \mu_l)$ with distinct parts, let $\nu_i = 1 + \mu_i$, taking $\mu_l = 0$ if necessary to make the number of parts even, the Schubert class for type D indexed by $w(\nu)$ is the image of $P_\mu(X)$ under the homomorphism in (5.4).

In order to do intersection computations with Schubert classes for G/B one could use the transition equations to expand any Schubert class in a unique

Schubert polynomials are defined in [2] for each of the classical families of root systems as elements in a certain inverse system. These polynomials can also be uniquely defined as the solutions to certain divided difference equations as we now explain. First, we describe the subring of the formal power series ring which contains the Schubert polynomials. Let $Q[z_1, z_2, \dots]$ be the ring of polynomials in countably many variables z_i. Although the number of variables is infinite, a polynomial has only a finite number of terms. Let $p_k(Z) = z_1^k + z_2^k + \cdots$ denote the k-th power sum of the z_i's—a formal power series, not a polynomial. Then $Q[z_1, z_2, \dots ; p_1(Z), p_3(Z), \dots]$ is the ring of formal power series which are polynomials in the z_i and the $p_k(Z)$ (k odd). These are all algebraically independent, so $\mathbb{Q}[z_1, z_2, \dots ; p_1(Z), p_3(Z), \dots]$ can also be regarded simply as the polynomial ring in variables z_i and p_k.

Definition. For each classical family, *Schubert polynomials*, S_w, are the unique homogeneous polynomials in $\mathbb{Q}[z_1, z_2, \dots]$ if $w \in S_\infty$ (or $\mathbb{Q}[z_1, z_2, \dots ; p_1, p_3, \dots]$ if $w \in B_\infty$ or $w \in D_\infty$) satisfying the equations

$$\partial_i S_w = \begin{cases} S_{w\sigma_i} & \text{if } l(w\sigma_i) < l(w) \\ 0 & \text{if } l(w\sigma_i) > l(w), \end{cases} \tag{4.9}$$

for all appropriate[2] divided difference operators, together with the condition that the constant term of S_w is 1 if $w = 1$ and 0 otherwise.

5. Transition equations

In this section we give the two main theorems. First, we remind the reader of Lascoux and Schützenberger's transition equation for Schubert polynomials of type A. Then, we give analogous transition equations for Schubert classes for the classical groups other than SL_n. These transition equations can be used to find the Schubert polynomials originally given in [14] or [2].

Proposition 3 (Lascoux and Schützenberger). [16, 4.16] *The Schubert polynomials of type A satisfy the following recursion formula:*

$$\mathfrak{S}_w = z_r \mathfrak{S}_v + \sum_{\substack{i<r \\ l(w)=l(vt_{ir})}} \mathfrak{S}_{vt_{ir}} \tag{5.1}$$

where r is the last descent of w, s is the largest index such that $w_s < w_r$, $v = wt_{rs}$, and $\mathfrak{S}_{id} = 1$.

[2]Equation (4.9) holds for each ∂_i such that σ_i is a generator of W_∞. Note that in types B, C, or D the equations hold for all $i \geq 0$ using ∂_0^B, ∂_0^C or ∂_0^D respectively.

Here α_i is considered to be the linear form given by replacing e_i by z_i, e.g. $\alpha_1 = z_2 - z_1$, and the inner product on linear forms is given by $(z_i, z_j) = \delta_{ij}$.

The *divided difference operators* [1, 5] ∂_i act on $f \in \mathbb{Q}[z_1, z_2, \ldots, z_n]$ by

$$\partial_i f = \frac{f - \sigma_i f}{-\alpha_i}. \tag{4.5}$$

Proposition 1. [1] *Define* $\partial_w = \partial_{a_1}\partial_{a_2}\cdots\partial_{a_p}$ *for any* $a_1 a_2 \cdots a_p \in R(w)$. *Then the operator* ∂_w *is independent of the choice of reduced word. The composition* $\partial_{a_1}\cdots\partial_{a_p}$ *is equal to 0 unless* $a_1 \cdots a_p$ *is reduced.*

It is a well-known theorem due to Bruhat that any classical Lie group G can be decomposed into the disjoint union of double cosets BwB for all w in the Weyl group of G, where B is a Borel subgroup of G. [1] The *Schubert variety* $X_w \subset G/B$ is defined to be the closure of the cell BwB/B. Let $H^*(G/B, \mathbb{Q})$ be the cohomology ring of G/B. Then each Schubert variety X_w determines a class in $H^*(G/B, \mathbb{Q})$ denoted $[X_w]$. There exists a isomorphism of rings

$$\phi : H^*(G/B, \mathbb{Q}) \to \frac{\mathbb{Q}[z_1, z_2, \ldots, z_n]}{I} \tag{4.6}$$

where I is the ideal generated by the homogeneous W-invariant polynomials without constant term. The Schubert classes S_w are defined to be the images of the classes of Schubert varieties under this map, *i.e.*,

$$S_w = \phi[X_{w_0 w}] \in \frac{\mathbb{Q}[z_1, z_2, \ldots, z_n]}{I}. \tag{4.7}$$

where w_0 is the unique element of W_n of longest length. Note, the divided difference operators are well defined on the quotient ring $\frac{\mathbb{Q}[z_1, z_2, \ldots, z_n]}{I}$ since $\partial_i(f)$ equals 0 for any $f \in I$.

Proposition 2. *The Schubert classes have the following three properties:*

1. *The set of Schubert classes* $\{[X_w] : w \in W\}$ *is a vector-space basis for* $H^*(G/B, \mathbb{Q})$ [4].
2. *Each* S_w *is homogeneous of degree* $l(w)$, *and if* w *is the identity element then* $S_w = 1$ [4].
3. *If* f *is a homogeneous element of degree* d *in* $\frac{\mathbb{Q}[z_1, z_2, \ldots, z_n]}{I}$ *then* [1, 5]

$$f = \sum_{l(w)=d} \partial_w(f) S_w. \tag{4.8}$$

[1] Harish-Chandra extended this theorem to semi-simple Lie groups.

	$w_i > 0, w_j > 0$	$w_i > 0, w_j < 0$	$w_i < 0, w_j > 0$	$w_i < 0, w_j < 0$
$\|w_i\| < \|w_j\|$	$1 + 2 \cdot U(w)$	$-1 - 2 \cdot U(ws_{ij})$	$1 + 2 \cdot U(w)$	$-1 - 2 \cdot U(ws_{ij})$
$\|w_i\| > \|w_j\|$	$1 + 2 \cdot U(w)$	$1 + 2 \cdot U(w)$	$-1 - 2 \cdot U(ws_{ij})$	$-1 - 2 \cdot U(ws_{ij})$

TABLE 3. Table of $l(ws_{ij}) - l(w)$ for $w \in D_\infty$ and $i < j$

We conclude this section with a lemma which we will use in proving Theorem 4. We say a signed permutation has a *descent* at position k if $w_k > w_{k+1}$. Recall that we consider barred numbers to be negative, so $w = 4\bar{1}\bar{6}3\bar{5}2$ has descents at positions 1, 2, and 4.

Lemma 3. *Given indices $i < j$ and $w \in W$ such that $l(ws_{ij}) = l(w) + 1$, then the descent set of w contains the descent set of ws_{ij}.*

Proof. The descent sets of w and ws_{ij} agree except possibly in positions $i-1$, i, $j-1$ and j. First, we eliminate the case $i = j - 1$ by noting $w_i < w_j$ if and only if $\overline{w_j} < \overline{w_i}$. By Lemma 2, it cannot happen that $-w_j < w_{i-1} < w_i$ or $-w_i < w_{j-1} < w_j$ so no descents can be created in positions $i-1$ or $j-1$. Again by Lemma 2, we have $w_j > -w_i$ (and $w_i > -w_j$), so if $w_j < w_{j+1}$ (or $w_i < w_{i+1}$) then $-w_i < w_{j+1}$ (respectively $-w_j < w_{i+1}$). Therefore, no new descents can be created in positions i or j either. □

4. Review of Schubert classes and Schubert polynomials

In this section we give the definitions and theorems from the literature that we will use in Section 5. We briefly review the flag manifold G/B in order to define the Schubert classes. The geometry of G/B implies that the Schubert classes in the cohomology ring of G/B have three properties which we will use to prove Theorem 4. Finally, we give the abstract definition for Schubert polynomials for the classical groups.

The Weyl groups act on polynomials $f \in \mathbb{Q}[z_1, z_2, \ldots, z_n]$ by

$$\sigma_i f(\ldots, z_i, z_{i+1}, \ldots) = f(\ldots, z_{i+1}, z_i, \ldots) \quad (i \geq 1), \tag{4.1}$$

$$\sigma_0 f(z_1, z_2, \ldots) = f(-z_1, z_2, \ldots), \tag{4.2}$$

$$\sigma_{\hat{1}} f(z_1, z_2, \ldots) = f(-z_2, -z_1, \ldots). \tag{4.3}$$

This action can also be described by extending the following action on the variables z_i to any polynomial:

$$\sigma_i z_k = z_k - \frac{2(z_k, \alpha_i)}{(\alpha_i, \alpha_i)} \alpha_i. \tag{4.4}$$

and no k exists such that

- $i < k < j$ *and* $w_i < w_k < w_j$.

The conditions in Lemma 1 characterize the covering relation for the Bruhat-Chevalley order on S_n completely. We state similar conditions for when $l(ws_{ij}) = l(w) + 1$ in Lemma 2. These conditions, along with Monk's conditions completely characterize the covering relation for the Bruhat-Chevalley order on the Weyl groups B_∞ and D_∞.

Lemma 2. *If $w \in B_\infty$ or D_∞, and $i \leq j$, then $l(ws_{ij}) = l(w) + 1$ if and only if*

- $-w_i < w_j$ *and* $-w_j < w_i$
- *in the case $w \in B_\infty$ and $i \neq j$, either $w_i < 0$ or $w_j < 0$,*

and no k exists such that either of the following are true:

- $k < i$ *and* $-w_j < w_k < w_i$
- $k < j$ *and* $-w_i < w_k < w_j$.

We omit a formal proof of Lemma 2 and instead refer the reader to Tables 1-3 below. Tables 1-3 give formulas for $l(ws_{ij}) - l(w)$. It is routine to verify the equations appearing in the tables by working out simple examples. For example take $w \in B_\infty$. Assume $i < j$, $w_i = 3$ and $w_j = 7$, then ws_{ij} has two additional negative signs plus the type I inversion at $(i < j)$ adding 3 to the length. Then for each combination of $k < i$, $i < k < j$ and $w_k = \pm 1, \pm 4, \pm 8$ determine the net contribution to $l(ws_{ij}) - l(w)$ from $(k < i)$, $(k < j)$, $(i < k)$ with $k < j$ to be

$$3+2\#\{k < i : -w_j < w_k < w_i\} + 2\#\{k < j : \ -w_i < w_k < w_j\}. \tag{3.3}$$

In general, let

$$U(w) = \#\{k < i : -w_j < w_k < w_i\} + \#\{k < j : \ -w_i < w_k < w_j\}.$$

	$w_i > 0, w_j > 0$	$w_i > 0, w_j < 0$	$w_i < 0, w_j > 0$	$w_i < 0, w_j < 0$
$\lvert w_i\rvert < \lvert w_j\rvert$	$3 + 2 \cdot U(w)$	$-1 - 2 \cdot U(ws_{ij})$	$1 + 2 \cdot U(w)$	$-3 - 2 \cdot U(ws_{ij})$
$\lvert w_i\rvert > \lvert w_j\rvert$	$3 + 2 \cdot U(w)$	$1 + 2 \cdot U(w)$	$-1 - 2 \cdot U(ws_{ij})$	$-3 - 2 \cdot U(ws_{ij})$

TABLE 1. Table of $l(ws_{ij}) - l(w)$ for $w \in B_\infty$ and $i < j$.

$w_i > 0$	$w_i < 0$
$1 + 2\#\{k < i : \ \lvert w_k\rvert < \lvert w_i\rvert\}$	$-1 - 2\#\{k < i : \ \lvert w_k\rvert < \lvert w_i\rvert\}$

TABLE 2. Table of $l(ws_{ii}) - l(w)$ for $w \in B_\infty$

Let $I(w)$ be the set of all positive roots $\alpha \in R$ for which $w(\alpha)$ is negative. $I(w)$ is called the *set of inversions* of w. The *length* of w (denoted $l(w)$) is the number of inversions of w. It is well known that $l(w)$ is also equal to the minimal number of generators needed to write w, [3, Ch. 4, Cor.2, p. 158]. If w can be written as a product of generators $\sigma_{a_1} \cdots \sigma_{a_p}$ and $p = l(w)$ then the sequence $a_1 \cdots a_p$ is a *reduced word* for w. Let $R(w)$ be the set of all reduced words for w.

One can analyze $I(w)$ in order to give formulas for $l(w)$. For example, consider the Weyl groups is of type B. If $0 < a < n$ and $w_a < 0$, then $e_a \in R_+$ and $w(e_a) = -e_{|w_a|} \in R_-$ so $e_a \in I(w)$. If $a < b$ and $|w_a| > |w_b|$ either $w(e_b - e_a) \in R_-$ or $w(e_b + e_a) \in R_-$ depending if $w_a < 0$ or $w_a > 0$, so either $e_b - e_a \in I(w)$ or $e_b + e_a \in I(w)$ (not both). If $a < b$, $w_b < 0$ and $|w_a| < |w_b|$ then $w(e_b - e_a) \in R_-$ and $w(e_b + e_a) \in R_-$, so both $e_b - e_a \in I(w)$ and $e_b + e_a \in I(w)$. All other positive roots remain positive under the action of w. Therefore, for $w \in B_\infty$,

(3.1)
$$l(w) = \#\{w_a < 0\} + \#\{a < b : \ |w_a| > |w_b|\} + 2\#\{a < b : w_b < 0 \text{ and } |w_a| < |w_b|\}.$$

Similarly, for $w \in D_\infty$ one can show

(3.2)
$$l(w) = \#\{a < b : \ |w_a| > |w_b|\} + 2\#\{a < b : \ w_b < 0 \text{ and } |w_a| < |w_b|\}.$$

The *reflections* in a Weyl group are the elements which correspond to a reflection with respect to a positive root. For the root systems of type A, B, C and D we will use the notation t_{ij} for the reflection with respect to $e_j - e_i$, s_{ij} for the reflection with respect to $e_i + e_j$ and s_{ii} for the reflection with respect to e_i or $2e_i$. In terms of signed permutations, t_{ij} is the transposition which switches positions i and j when acting on the right, and s_{ij} switches positions i and j and the sign of the elements in those positions when acting on the right. Note that s_{ii} just switches the sign on position i. The only reflections in S_∞ are of the form t_{ij} and the only reflections in D_∞ are of the form t_{ij} and s_{ij} for $i < j$.

Definition. The *Bruhat-Chevalley* order on elements of a Weyl group is defined by the covering relation $w \to v$ if and only if given any $a_1 \cdots a_p \in R(v)$ there exists an index r such that $a_1 \cdots \hat{a_r} \cdots a_p \in R(w)$. Equivalently, $w \to v$ if there exists a reflection t such that $v = wt$ and $l(v) = l(w) + 1$ [4, Proposition 6].

Lemma 1 (Monk, [17]**).** *If $w \in S_\infty, B_\infty$ or D_∞, and $i < j$, then $l(wt_{ij}) = l(w) + 1$ if and only if*

- $w_i < w_j$

These bases are slightly different than those stated in [10] or [15]. We have changed the bases so that the basis of the root system R_i is contained in the basis of R_{i+1} in each case.

For each root system R, we partition the set of all roots into *positive roots* R_+ and *negative roots* R_-. For each $\alpha \in R$, let $\alpha = \sum_{\beta \in B} m_\beta \beta$. If all the coefficients m_β are greater than or equal to 0, then $\alpha \in R_+$, and if all m_β are less than or equal to 0 then $\alpha \in R_-$. For each family of root systems A, B, C, and D, the positive roots with respect to the basis in (2.2) are given by

(2.3)

$$\begin{aligned}
A_{n-1} : R_+ =& \{e_j - e_i : 1 \le i < j \le n\} \\
B_n : R_+ =& \{e_j - e_i : 1 \le i < j \le n\} \cup \{e_i + e_j : \ 1 \le i < j \le n\} \cup \{e_i : \ 1 \le i \le n\} \\
C_n : R_+ =& \{e_j - e_i : 1 \le i < j \le n\} \cup \{e_i + e_j : \ 1 \le i < j \le n\} \cup \{2e_i : \ 1 \le i \le n\} \\
D_n : R_+ =& \{e_j - e_i : 1 \le i < j \le n\} \cup \{e_i + e_j : \ 1 \le i < j \le n\}
\end{aligned}$$

Let S_n be the symmetric group whose elements are permutations written in one-line notation as $[w_1, w_2, \ldots, w_n]$. S_n is the Weyl group of A_{n-1} so it is generated by the adjacent transpositions σ_i for $1 \le i < n$, where σ_i interchanges positions i and $i+1$ when acting on the right, *i.e.,* $[\ldots, w_i, w_{i+1}, \ldots]\sigma_i = [\ldots, w_{i+1}, w_i, \ldots]$.

Let B_n be the hyperoctahedral group (or signed permutation group). B_n is the Weyl group for both root systems B_n and C_n (pardon the abuse of notation). The elements of B_n are permutations with a sign attached to every entry. We use the compact notation where a bar is written over an element with a negative sign. For example $[\bar{3}, 2, \bar{1}] \in B_3$. B_n is generated by the adjacent transpositions σ_i for $1 \le i < n$, as in S_n, along with σ_0 which acts on the right by changing the sign of the first element, *i.e.,* $[w_1, w_2, \ldots, w_n]\sigma_0 = [\overline{w_1}, w_2, \ldots, w_n]$.

Let D_n be the subgroup of B_n consisting of all signed permutations with an even number of sign changes. D_n is the Weyl group of the root system D_n so it is generated by σ_i for $1 \le i < n$ and the additional generator $\sigma_{\hat{1}}$ where $[w_1, w_2, \ldots, w_n]\sigma_{\hat{1}} = [\overline{w_2}, \overline{w_1}, w_3, \ldots, w_n]$.

Each of the Weyl groups W_n can be included in W_{n+1} as the subgroup of elements w with $w_{n+1} = n+1$. In fact, each family of Weyl groups above can be extended to an infinite group $W_\infty = \bigcup_n W_n$.

3. Reflections in a Weyl group

Let R be a root system of type A, B, C, or D, and let W be its associated Weyl group. Each $w \in W$ acts on R by acting on the underlying vector space V. Using the same bases as (2.2) and our notation $w = [w_1, \ldots, w_n]$, then $w(e_i) = e_{w_i}$, where we will use the convention $e_{\bar{i}} = -e_i$.

computing $l(w\sigma_\beta) - l(w)$ where σ_β is a reflection in the Weyl group. In particular, it will be easy to say precisely when $l(w\sigma_\beta) = l(w) + 1$, characterizing the covering relations for the Bruhat-Chevalley order. In Section 4, divided difference operators, Schubert classes, and Schubert polynomials are defined. Finally, in Section 5 the main theorems and their corollaries are given.

2. Notation on root systems

Let V be any finite dimensional vector space over $\mathbb{Q}$ with a positive definite symmetric bilinear form (α, β). Each vector $\alpha \in V$ determines a *reflection* σ_α which fixes the hyperplane perpendicular to α and sends α to $-\alpha$. Formally,

$$\sigma_\alpha(v) = v - \frac{2(v,\alpha)}{(\alpha,\alpha)}\alpha. \tag{2.1}$$

A subset R of V is called a *root system* [10] if

(1) R is finite and $0 \notin R$.
(2) If $\alpha \in R$, then σ_α leaves R invariant.
(3) If $\alpha \in R$ then $-\alpha$ is the only other scalar multiple of α in R.
(4) If $\alpha, \beta \in R$ then $\frac{2(\alpha,\beta)}{(\beta,\beta)} \in \mathbb{Z}$.

The *Weyl group* W associated with R is the group generated by the reflections σ_α where $\alpha \in R$. A *basis* of R is a subset B of R such that

(1) B is linearly independent.
(2) For each $\alpha \in R$, $\alpha = \sum_{\beta \in B} m_\beta \beta$ with coefficients $m_\beta \in \mathbb{Z}$ and either all $m_\beta \geq 0$ or all $m_\beta \leq 0$.

If B is a basis of R and W is the associated Weyl group then W is generated by $(\sigma_\beta : \beta \in B)$.

Reduced irreducible root systems can be completely classified to be one of 9 types: four infinite families A_n, B_n, C_n, D_n for any positive integer n, and five exceptional root systems E_6, E_7, E_8, F_4, G_2, [15]. Our discussion only includes the infinite families of root systems, these are the root systems which correspond with the classical Lie groups. Lists of bases and Weyl group generators can be found in [10] for the exceptional root systems.

Fix an orthonormal basis $\{e_1, ..., e_n\}$ for the vector space V. With respect to this basis, we take the following vectors as bases for the classical root systems:

$$\begin{aligned}
A_{n-1} : B &= \{e_{i+1} - e_i : 1 \leq i \leq n-1\} \\
B_n : B &= \{e_1\} \cup \{e_{i+1} - e_i : 1 \leq i \leq n-1\} \\
C_n : B &= \{2e_1\} \cup \{e_{i+1} - e_i : 1 \leq i \leq n-1\} \\
D_n : B &= \{e_1 + e_2\} \cup \{e_{i+1} - e_i : 1 \leq i \leq n-1\}
\end{aligned} \tag{2.2}$$

Lascoux and Schützenberger to facilitate computations of intersection multiplicities for Schubert varieties. For a good survey on Schubert polynomials for SL_n consult [16]. In [2], the definition of Schubert polynomials has been extended from SL_n to the other classical groups. There, Schubert polynomials are defined to be elements in an inverse system. These polynomials satisfy an infinite system divided difference equations and are stable under inclusion into an infinite union of Weyl groups (recently other possible definitions have been given by [8]).

The first theorem of this paper gives new recursive formulas, called *transition formulas* for the Schubert classes of G/B, in the case G is one of the classical Lie groups, $Sp(2n,\mathbb{C})$, $SO(2n,\mathbb{C})$, and $SO(2n+1,\mathbb{C})$. For example, the transition formula for $Sp(2n)$ implies

$$S_{\overline{6}\overline{1}5\overline{4}\overline{2}3} = z_3 S_{\overline{6}\overline{1}3\overline{4}\overline{2}5} + S_{\overline{6}3\overline{1}\overline{4}\overline{2}5} + S_{\overline{6}\overline{3}1\overline{4}\overline{2}5} + S_{\overline{6}\overline{1}2\overline{4}\overline{3}5}, \tag{1.1}$$

(see Theorem 4). This formula is the analog of the transition equation given by Lascoux and Schützenberger, see [16, p.52] for details. One can use this formula along with a theorem due to Pragacz[18] to obtain representatives for the Schubert classes. The second theorem states that the representatives obtained in this manner from the transition equations are exactly the Schubert polynomials found in [2]. This approach to computing Schubert polynomials has the advantage that no explicit computations with divided difference operators are necessary. The transition equations provide one of the most efficient methods of computing any of the Schubert polynomials of types A, B, C, and D. As a corollary to the theorem on transition equations, we also give a new recurrence formula for the Stanley symmetric functions.

Sometime in the late 1950's (finally published in 1994), Chevalley [4] gave a formula for intersecting a Schubert variety of minimal codimension, X_{σ_r}, with any other Schubert variety X_w in general position. For Schubert varieties in SL_n/B this formula is typically attributed to Monk [17]. It is a general consequence of the theory of Schubert classes (or polynomials) that intersection formulas on Schubert varieties can be stated in terms of products of Schubert classes (or polynomials). As corollaries to the main theorems we give a new proof of Chevalley's product formula and restate this product formula in the style of Monk's formula for intersections of Schubert varieties in SL_n/B.

Due to the fact that Chevalley's paper went unpublished, his results are often attributed to other people. In addition to giving Monk's formula contemporaneously with Monk, Chevalley apparently was the first to define what is now called the Bruhat ordering of the elements in a Weyl group of a semisimple Lie group, see Proposition 6 of [4].

We begin with a brief review of root systems and Weyl groups and discuss reflections in an arbitrary Weyl group. In Tables 1–3 we give formulas for

TRANSITION EQUATIONS FOR ISOTROPIC FLAG MANIFOLDS

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ABSTRACT. In analogy with transition equations for type A Schubert polynomials given by Lascoux and Schützenberger's [14], we give recursive formulas for computing representatives of the Schubert classes for the isotropic flag manifolds. These representatives are exactly the Schubert polynomials found in [2]. This new approach to finding Schubert polynomials is very closely related to the geometry of the flag manifold and has the advantage that it does not require explicit computations with divided difference operators. The generalized transition equations also lead to a recursion for Stanley symmetric functions and a new proof of Chevalley's intersection formula for Schubert varieties. The proofs involve a careful study of the Bruhat order for the Weyl groups and two simple lemmas for applying divided difference operators.

1. INTRODUCTION

Schubert polynomials are explicit representatives for the Schubert classes studied by Bernstein, Gelfand, and Gelfand [1], Demazure [5] and others. In the context of SL_n, these polynomials were invented and carefully studied by

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