

LOWER BOUNDS FOR KAZHDAN–LUSZTIG POLYNOMIALS FROM PATTERNS

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Abstract. Kazhdan–Lusztig polynomials $P_{x,w}(q)$ play an important role in the study of Schubert varieties as well as the representation theory of semisimple Lie algebras. We give a lower bound for the values $P_{x,w}(1)$ in terms of “patterns”. A pattern for an element of a Weyl group is its image under a combinatorially defined map to a subgroup generated by reflections. This generalizes the classical definition of patterns in symmetric groups. This map corresponds geometrically to restriction to the fixed point set of an action of a one-dimensional torus on the flag variety of a semisimple group G . Our lower bound comes from applying a decomposition theorem for “hyperbolic localization” [Br] to this torus action. This gives a geometric explanation for the appearance of pattern avoidance in the study of singularities of Schubert varieties.

1. Introduction

Many recent results on the singularities of Schubert varieties X_w in the variety \mathcal{F}_n of flags in \mathbb{C}^n are expressed by the existence of certain patterns in the indexing permutation $w \in \mathfrak{S}_n$. For example, Lakshmibai and Sandhya [LS] proved that X_w is singular if and only if w contains either of the patterns 4231 or 3412 (see also [R] and [W]). A permutation $w \in \mathfrak{S}_n$ is said to contain the pattern $\tilde{w} \in \mathfrak{S}_k$ for $k < n$ if the permutation matrix of w has the permutation matrix of \tilde{w} as a submatrix.

This implies that if $\tilde{w} \in \mathfrak{S}_k$ is any pattern for w and $X_{\tilde{w}} \subset \mathcal{F}_k$ is singular, then X_w is singular as well. In this paper, we give a general geometric explanation of this phenomenon which works for the flag variety \mathcal{F} and Weyl group W of any semisimple algebraic group G .

Our result concerns the Kazhdan–Lusztig polynomials $P_{x,w}(q) \in \mathbb{Z}_{\geq 0}[q]$, $x, w \in W$. Although defined purely combinatorially, they carry important information about representation theory of Hecke algebras and Lie algebras (see [KL1, BB, BryK, BGS] among many others), as well as geometric information about the singularities of Schubert varieties X_w in \mathcal{F} .

More precisely, $P_{x,w}(q)$ is the Poincaré polynomial (in $q^{1/2}$) of the local intersection cohomology of X_w at a generic point of X_x , and $P_{1,w}(1) = 1$ if and only if X_w is

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rational smooth [KL2]. If G is of type **A**, **D** or **E**, then X_w is singular if and only if $P_{1,w}(1) > 1$ (Deodhar [De] proved this for type **A**, while Peterson (unpublished) proved it for all simply laced groups. See [CK]).

Our main result (Theorem 4) is a lower bound for $P_{x,w}(1)$ in terms of Kazhdan–Lusztig polynomials of patterns appearing in x and other elements of W determined by x and w . Here a pattern of an element of W is its image under a function $\phi: W \rightarrow W'$, which we define for any finite Coxeter group and any (not necessarily standard) parabolic subgroup $W' \subset W$. It agrees with the standard definition of patterns in type **A**, but is more general than the one using signed permutations used in [Bi] for types **B** and **D**.

One consequence of our result is the following:

Theorem 1. *For any parabolic $W' \subset W$, we have $P_{1,w}(1) \geq P_{1,\phi(w)}(1)$.*

In particular, this gives another proof that $X_{\bar{w}}$ singular implies X_w singular in type **A**. See also the remark after Theorem 10.

The definition of the pattern map ϕ is combinatorial, but it is motivated by the geometry of the action of the torus T on \mathcal{F} , and the proof of Theorem 4 is entirely geometrical. For $W' \subset W$ parabolic, there is a cocharacter $\rho: \mathbb{C}^* \rightarrow T$ whose fixed point set in \mathcal{F} is a disjoint union of copies of the flag variety \mathcal{F}' of a group G' with Weyl group W' . The action of ρ gives rise to a “hyperbolic localization” functor which takes sheaves on \mathcal{F} to sheaves on \mathcal{F}' . Theorem 4 then follows from a “decomposition theorem” for this functor, proved in [Br], together with the fact that hyperbolic localization preserves local Euler characteristics.

If the action is totally attracting or repelling near a fixed point, hyperbolic localization is just ordinary restriction or its Verdier dual. This gives stronger coefficient-by-coefficient inequalities in some special cases (see Theorem 5). The attracting/repelling case of [Br] has been known for some time; it was used in [BrM] to prove a conjecture of Kalai on toric g -numbers of rational convex polytopes.

Matthew Dyer has recently given us a preprint [Dy] containing an inequality equivalent to Theorem 4, which he proves using his theory of abstract highest weight categories. It seems likely that his approach is dual to ours under some version of Koszul duality [BGS].

This work was originally motivated by the following question asked by Francesco Brenti: How can we describe the Weyl group elements w such that $P_{\text{id},w}(1) = 2$? In type **A**, we can show that if $P_{\text{id},w}(1) = 2$, then the singular locus of the Schubert variety X_w has only one irreducible component and w must avoid the patterns:

$$\begin{array}{ccc} (526413) & (546213) & (463152) \\ (465132) & (632541) & (653421) \end{array}$$

We conjecture the converse holds as well.

We outline the sections of this paper. In §2.1, we discuss pattern avoidance on permutations and some applications from the literature. In §2.2 we describe the pattern map for arbitrary finite Coxeter groups. §2.3 explains why the two notions agree for permutations. The main result of §2.2 is proved in §2.4. In §3 we state our main theorem. In §3.1 we highlight two particularly interesting special cases, including Theorem 1. Our geometric arguments are in §4.

2. Pattern avoidance

2.1. Classical pattern avoidance

We can write an element w of the permutation group \mathfrak{S}_n on n letters in a one-line notation as $w = w_1w_2 \cdots w_n$, i.e., w maps i to w_i . We say a permutation w *contains* a pattern $v \in \mathfrak{S}_k$ if there exists a subsequence $w_{i_1}w_{i_2} \cdots w_{i_k}$, with the same relative order as $v = v_1 \cdots v_k$. If no such subsequence exists we say w *avoids* the pattern v .

More formally, let $a_1 \cdots a_k$ be any list of distinct positive integers. Define the *flattening function* $\text{fl}(a_1 \cdots a_k)$ to be the unique permutation $v \in \mathfrak{S}_k$ such that $v_i > v_j \iff a_i > a_j$. Then it is equivalent to say that w *avoids* v if no $\text{fl}(w_{i_1}w_{i_2} \cdots w_{i_k}) = v$. For example, $w = 4536172$ contains the pattern 3412, since $\text{fl}(w_1w_4w_5w_7) = \text{fl}(4612) = 3412$, but it avoids 4321.

Several properties of permutations have been characterized by pattern avoidance and containment. For example, as mentioned in the introduction, for the Schubert variety X_w we have Schubert variety X_w is nonsingular if and only if $P_{1,w} = 1$ if and only if w avoids 3412 and 4231 [LS, C, De, KL2]. The element C'_w of the Kazhdan–Lusztig basis of the Hecke algebra of W equals the product $C'_{s_{a_1}} C'_{s_{a_2}} \cdots C'_{s_{a_p}}$ for any reduced expression $w = s_{a_1}s_{a_2} \cdots s_{a_p}$ if and only if w is *321-hexagon-avoiding*, [BiW1]. Here 321-hexagon-avoiding means w avoids the five patterns 321, 56781234, 46781235, 56718234, 46718235.

The notion of pattern avoidance easily generalizes to the Weyl groups of types B,C,D since elements can be represented in one-line notation as permutations with \pm signs on the entries. Once again, the properties $P_{1,w} = 1$ and $C'_w = C'_{s_{a_1}} C'_{s_{a_2}} \cdots C'_{s_{a_p}}$ can be characterized by pattern avoidance [Bi, ?], though the list of patterns can be rather long. More examples of pattern avoidance appear in [LasSc, St, BiP, BiW2, Ma, KLR, Co, Co2].

2.2. Patterns in Coxeter groups

In this section, we generalize the flattening function for permutations to an arbitrary finite Coxeter group W .

Let S be the set of simple reflections generating W . The set R of all reflections is $R = \bigcup_{w \in W} wSw^{-1}$. Given $w \in W$, its length $l(w)$ is the length of the shortest expression for w in terms of elements of S . The Bruhat–Chevalley order is the partial order \leq on W generated by the relation

$$x < y \text{ if } l(x) < l(y) \text{ and } xy^{-1} \in R.$$

Each subset $I \subset S$ generates a subgroup W_I ; a subgroup $W' \subset W$ which is conjugate to W_I for some I is called a *parabolic* subgroup. The W_I 's themselves are known as *standard* parabolic subgroups.

A parabolic subgroup $W' = xW_Ix^{-1}$ of W is again a Coxeter group, with simple reflections $S' = xIx^{-1}$ and reflections $R' = R \cap W'$. Note that $S' \not\subset S$ unless W' is standard.

We denote the length function and the Bruhat–Chevalley order for (W', S') by l' and \leq' , respectively. If $W' = W_I$, then

$$l' = l|_{W'} \text{ and } \leq' = \leq|_{W' \times W'},$$

but in general we only have $l'(w) \leq l(w)$ and $x \leq' y \implies x \leq y$. For instance, if $W' \subset \mathfrak{S}_4$ is generated by the reflections $r_{23} = 1324$ and $r_{14} = 4231$, then $r_{23} \leq r_{14}$ although they are not comparable for \leq' .

The following theorem/definition generalizes the flattening function for permutations.

Theorem 2. *Let $W' \subset W$ be a parabolic subgroup. There is a unique function $\phi: W \rightarrow W'$, the pattern map for W' , satisfying the following:*

- (a) ϕ is W' -equivariant: $\phi(wx) = w\phi(x)$ for all $w \in W', x \in W$,
- (b) If $\phi(x) \leq' \phi(wx)$ for some $w \in W'$, then $x \leq wx$.

In particular, ϕ restricts to the identity map on W' .

If $W' = W_I$ is a standard parabolic, then (b) can be strengthened to “if and only if”. In this case the result is well known.

To show uniqueness, note that (a) implies that ϕ is determined by the set $\phi^{-1}(1)$, and (b) implies that $\phi^{-1}(1) \cap W'x$ is the unique minimal element in $W'x$. Existence is more subtle; it is not immediately obvious that the function so defined satisfies (b). We give a construction of a function ϕ that satisfies (a) and (b) in Section 2.4.

2.3. Relation with classical patterns

Take integers $1 \leq a_1 < \dots < a_k \leq n$, and let $\Sigma = \{a_1, a_2, \dots, a_k\}$. Define a generalized flattening function $\text{fl}_\Sigma: \mathfrak{S}_n \rightarrow \mathfrak{S}_k$ by $\text{fl}_\Sigma(w) = \text{fl}(w_{i_1}w_{i_2} \dots w_{i_k})$, where $w_{i_j} \in \Sigma$ for all $1 \leq j \leq k$ and $1 \leq i_1 < i_2 < \dots < i_k \leq n$.

Let $W' \subset \mathfrak{S}_n$ be the subgroup generated by the transpositions r_{a_i, a_j} for all $i < j$. It is parabolic; conjugating by any permutation z with $z_i = a_i$ for $1 \leq i \leq k$ gives an isomorphism $\iota: \mathfrak{S}_k \rightarrow W'$, where $\mathfrak{S}_k \subset \mathfrak{S}_n$ consists of permutations fixing the elements $k + 1, \dots, n$.

The function $\iota \circ \text{fl}_\Sigma$ satisfies the properties of Theorem 2, and so $\iota \circ \text{fl}_\Sigma(w) = \phi(w)$. Property (a) follows since left multiplication by a permutation $w \in W'$ acts only on the values in the set $\{a_1, a_2, \dots, a_k\}$. To prove (b), note that if $v_i = w_i$ for two permutations $v, w \in \mathfrak{S}_n$, then $v \leq w$ if and only if $\text{fl}(\hat{v}) \leq \text{fl}(\hat{w})$ where \hat{v}, \hat{w} are the sequences obtained by removing the i th entry from each. This implies that $\iota \circ \text{fl}_\Sigma(x) \leq' \iota \circ \text{fl}_\Sigma(wx)$ if and only if $x \leq wx$.

For example, take $\Sigma = \{1, 4, 6, 7\}$; the associated subgroup $W' \subset \mathfrak{S}_7$ is generated by $\{r_{14}, r_{46}, r_{67}\}$. If $x = 6213475$ then $y = 1243675$ is the unique minimal element in $W'x$ and $x = r_{46}r_{14}y$, so $\phi(x) = r_{46}r_{14}$. This agrees with the classical flattening using the isomorphism $W' \cong \mathfrak{S}_4$ given by $r_{14} \mapsto s_1, r_{46} \mapsto s_2, r_{67} \mapsto s_3$: in fact,

$$\text{fl}_{\{1,4,6,7\}}(6213475) = \text{fl}(6147) = 3124 = s_2s_1.$$

To obtain the most general parabolic subgroup of \mathfrak{S}_n , let $\Sigma_1, \dots, \Sigma_l$ be disjoint subsets of $1 \dots n$. To each Σ_j is associated a parabolic subgroup W'_j as before, and then

$$W' = W'_1W'_2 \dots W'_l \cong \mathfrak{S}_{|\Sigma_1|} \times \dots \times \mathfrak{S}_{|\Sigma_l|}$$

is a parabolic subgroup. The corresponding flattening function is

$$w \mapsto (\text{fl}_{\Sigma_1}(w), \dots, \text{fl}_{\Sigma_l}(w)).$$

In types **B** and **D**, the flattening function of [Bi] given in terms of signed permutations can also be viewed as an instance of our pattern map. The group W' of signed permutations which fix every element except possibly the $\pm a_i$, $1 \leq i \leq k$ is parabolic. Multiplication on the left by $w \in W'$ acts only on the values in the set $\{\pm a_1, \pm a_2, \dots, \pm a_k\}$ and if $v_i = w_i$ for two signed permutations v, w , then $v \leq w$ if and only if $\text{fl}(\hat{v}) \leq \text{fl}(\hat{w})$ where \hat{v}, \hat{w} are the sequences obtained by removing the i th entry from each. It follows that $v \mapsto \text{fl}(\hat{v})$ satisfies the conditions of Theorem 2

There are other types of parabolic subgroups in types **B** and **D** which give rise to other pattern maps. For instance, the group W' of all unsigned permutations is a parabolic subgroup of either B_n or D_n . In this case the pattern map “flattens” the signed permutation to an unsigned one (e.g., $-4, 2, 1, -3 \mapsto 1432$). Other cases of pattern maps for classical groups are more difficult to describe combinatorially.

The first author and Postnikov [BiP] have used these more general pattern maps to reduce significantly the number of patterns needed to recognize smoothness and rational smoothness of Schubert varieties. They reduce the list even further by generalizing pattern maps to the case of “root system embeddings” which do not necessarily preserve the inner products of the roots; for instance, there is a root system embedding of A_3 into B_3 . We do not know of a geometric interpretation of these more general pattern maps.

2.4. Spanning subgroups and the reflection representation

To prove Theorem 2 we use the action of W on its root system. See [H, Section 1] for proofs of the following facts.

We have the following data: a representation of W on a finite-dimensional real vector space V , a W -invariant subset $\Phi \subset V$ (the roots), a subset $\Pi \subset \Phi$ (the positive roots), and a bijection $r \mapsto \alpha_r$ between R and Π .

These data satisfy the following properties: Φ is the disjoint union of Π and $-\Pi$. The vectors $\{\alpha_s\}_{s \in S}$ form a basis for V ; a root $\alpha \in \Phi$ is positive if and only if it can be expressed in this basis with nonnegative coefficients. For any $r \in R$ and $w \in W$, we have

$$rw > w \iff \alpha_r \in w\Pi. \tag{1}$$

Given a linear function $H: V \rightarrow \mathbb{R}$, define

$$\Pi_H = \{\alpha \in \Phi \mid H(\alpha) > 0\}.$$

Call H *generic* if $\Phi \cap \ker H = \emptyset$. If we take $H_1(\alpha_s) = 1$ for all $s \in S$, then H_1 is generic and $\Pi = \Pi_{H_1}$. If we put $H_w = H_1 \circ w^{-1}$, then $\Pi_{H_w} = w\Pi$. Conversely, if H is generic, then $\Pi_H = w\Pi$ for a unique $w \in W$.

Proposition 3. *Let $W' \subset W$ be a subgroup generated by reflections. Then W' is parabolic if and only if there is a subspace $V' \subset V$ so that the subgroup W' is generated by $R' = \{r \in R \mid \alpha_r \in V'\}$. If so, then V' is W' -stable, and putting $\Phi' = \Phi \cap V'$, $\Pi' = \Pi \cap V'$, and $\alpha'_r = \alpha_r$ for $r \in R'$ gives the reflection representation of W' .*

Proof. See [H, §1.12]. \square

Remark. In type **A**, all subgroups generated by reflections are parabolic. In other types this is no longer the case – for instance, the subgroup $W' \cong (\mathbb{Z}_2)^n$ of B_n generated by reflections in the roots $\{\pm e_j\}$ is not parabolic for any $n \geq 2$, since these roots span V .

We now prove the existence of the function ϕ from Theorem 2. Let $V' \subset V$ be as in Proposition 3. Given $w \in W$, we have $w\Pi = \Pi_{H_w}$, and so $w\Pi \cap V' = \Pi'_{H'}$, where $\Pi' = \Pi \cap V$ and $H' = H_w|_{V'}$. It follows that there is a unique $\phi(w) \in W'$ so that

$$\phi(w)\Pi' = w\Pi \cap V'.$$

We show that the function ϕ defined this way satisfies (a) and (b) from Theorem 2. Any $w \in W'$ fixes V' , so if $x \in W$, then

$$\phi(wx)\Pi' = (wx\Pi) \cap V' = w(x\Pi \cap V') = w\phi(x)\Pi',$$

giving (a).

To prove (b), it will be enough to show that $\phi(x) \leq' \phi(rx)$ implies $x \leq rx$ for any $x \in W$, $r \in R'$, since these relations generate the Bruhat–Chevalley orders on W and W' . We have

$$\begin{aligned} \phi(x) <' \phi(rx) = r\phi(x) &\iff \alpha_r \in \phi(x)\Pi' = x\Pi \cap V' \\ &\implies \alpha_r \in x\Pi \\ &\implies x < rx. \end{aligned}$$

3. The main result

Suppose now that W is the Weyl group of a semisimple complex algebraic group G . Let $W' \subset W$ be parabolic, and let $\phi: W \rightarrow W'$ be the pattern map of Theorem 2. For any $x \in W$, define a partial order on $W'x$ by “pulling back” the Bruhat order from W' : if $w, w' \in W'$, say $wx \leq_x w'x$ if and only if $\phi(wx) \leq' \phi(w'x)$. By Theorem 2, this is weaker than the Bruhat order on $W'x$.

Our main result is the following.

Theorem 4. *If $x, w \in W$, then*

$$P_{x,w}(1) \geq \sum_{y \in M(x,w;W')} P_{y,w}(1) P'_{\phi(x),\phi(y)}(1),$$

where $M(x,w;W')$ is the set of maximal elements with respect to \leq_x in $[1,w] \cap W'x$, and P' denotes the Kazhdan–Lusztig polynomial for the Coxeter system (W', S') .

Conjecturally this should hold for any finite Coxeter group W . There is a stronger formulation when W' is a standard parabolic subgroup of W ; see the next section.

Example. Take $W = \mathfrak{S}_4$, $w = 4231$, $x = 2143$. Let $W' \cong \mathfrak{S}_2 \times \mathfrak{S}_2$ be the group generated by reflections $r_{13} = 3214$, $r_{24} = 1432$. Then $W'x = \{2143, 4123, 2341, 4321\}$. All but 4321 are in the interval $[1,w]$, so the maximal elements of $[1,w] \cap W'x$ are $4123 = r_{24}x$ and $2341 = r_{13}x$. Theorem 4 gives

$$\begin{aligned} P_{2143,4231}(1) &\geq P_{4123,4231}(1)P'_{1,r_{24}}(1) + P_{2341,4231}(1)P'_{1,r_{13}}(1) \\ &= 1 \cdot 1 + 1 \cdot 1 = 2, \end{aligned}$$

which holds since $P_{2143,4231}(q) = 1 + q$.

Note that this shows X_{4231} is singular, even though all the Schubert varieties corresponding to terms on the right-hand side are smooth.

Example. One can calculate $P_{1234567,6734512}(1) = 44$ in type **A**. This is the maximum value of $P_{x,w}(1)$ for any $x, w \in \mathfrak{S}_7$. Let $W' \subset \mathfrak{S}_9$ be the subgroup generated by the reflections $\{r_{13}, r_{34}, r_{45}, r_{57}, r_{78}, r_{89}\}$; it is a parabolic subgroup isomorphic to \mathfrak{S}_7 . If $w = 869457213$ and $x = 163457289$, then $W'x = W'w$ so $M(x, w; W') = \{w\}$, giving $\phi(x) = 1234567$ and $\phi(w) = 6734512$. Hence

$$P_{x,w}(1) \geq P'_{1234567,6734512}(1)P_{w,w}(1) = 44.$$

3.1. Special cases/applications

The complicated interaction of the multiplicative structure of W and the Bruhat–Chevalley order makes computing the set $M(x, w; W')$ difficult. We mention two cases in which the answer is nice:

(a) If w and x lie in the same W' -coset, then $M(x, w; W') = \{w\}$. In this case Theorem 4 says

$$P_{x,w}(1) \geq P'_{\phi(x),\phi(w)}(1).$$

This allows us to prove Theorem 1 from the introduction: Given $w \in W$, let $x \in W'w$ satisfy $\phi(x) = 1$. Then

$$P_{1,w}(1) \geq P_{x,w}(1) \geq P'_{1,\phi(w)}(1).$$

The first inequality comes from the monotonicity of Kazhdan–Lusztig polynomials [I], [BrM2, Corollary 3.7].

(b) If either W' or $x^{-1}W'x$ is a standard parabolic subgroup of W , then $M(x, w; W')$ has only one element. The case where $x = 1$ was studied by Billey, Fan, and Losonczy [BiFL].

In this case the inequality will hold coefficient by coefficient rather than just at $q = 1$:

Theorem 5. *If W' or $x^{-1}W'x$ is a standard parabolic subgroup, then*

$$[q^k]P_{x,w} \geq \sum_{i+j=k} [q^i]P_{y,w} [q^j]P'_{\phi(x),\phi(y)},$$

where $M(x, w; W') = \{y\}$. Here the notation $[q^k]P$ means the coefficient of q^k in the polynomial P .

If both (a) and (b) hold, then Theorem 5 is implied by a well known equality (see [P, Lemma 2.6]):

Theorem 6. *If W' or $x^{-1}W'x$ is a standard parabolic subgroup of W and $w \in W'x$, then*

$$P_{x,w}(q) = P'_{\phi(x),\phi(w)}(q).$$

Theorem 6 can be thought of as a generalization of a theorem due to Brenti and Simion:

Theorem 7. [BreS] *Let $u, v \in \mathfrak{S}_n$. For any $1 \leq i \leq n$ such that $\{1, 2, \dots, i\}$ appear in the same set of positions (though not necessarily in the same order) in both u and v , then*

$$P_{u,v}(q) = P_{u[1,i],v[1,i]}(q) \cdot P_{\text{fl}(u[i+1,n]),\text{fl}(v[i+1,n])}(q),$$

where $u[j, k]$ is obtained from u by only keeping the numbers $j, j + 1, \dots, k$ in the order they appear in u .

We demonstrate the relationship between the two theorems in an example. Let $I_1 = \{s_1, s_2, s_3\}$, $I_2 = \{s_5, s_6, s_7\}$, $I = I_1 \cup I_2$. Let $W' = W_I \cong W_{I_1} \times W_{I_2}$. Any pair x, w in the same coset of $W' \setminus W$ satisfies the conditions of Theorem 7 and Theorem 6. Take $x = 25174683$ and $w = 48273561$. Then Theorem 6 gives

$$\begin{aligned} P_{25174683,48273561}(q) &= P'_{\phi(25174683),\phi(48273561)}(q) \\ &= P'_{21435768,42318756}(q) = P_{2143,4231}(q)P_{1324,4312}(q) \end{aligned}$$

agreeing with Theorem 7. The last equality results because we have $P_{x_1 \times x_2, w_1 \times w_2}(q) = P_{x_1, w_1}(q)P_{x_2, w_2}(q)$ for any $x_1 \times x_2, w_1 \times w_2$ in the reducible Coxeter group $W_{I_1} \times W_{I_2}$.

4. Geometry of flag varieties

Let G be a connected semisimple linear algebraic group over \mathbb{C} . It acts transitively on the flag variety \mathcal{F} of Borel subgroups of G by conjugation: $g \cdot B = gBg^{-1}$. For any $g \in G$, the point $B \in \mathcal{F}$ is fixed by g if and only if $g \in B$.

Fix a Borel subgroup and a maximal torus $T \subset B \subset G$. The Weyl group $W = N_G(T)/T$ is a finite Coxeter group. The point $g \cdot B \in \mathcal{F}$ is fixed by T if and only if $g \in N_G(T)B$, and so $g \mapsto g \cdot B$ induces a bijection between W and \mathcal{F}^T . We abuse notation and refer to $w \in W$ and the corresponding point of \mathcal{F} by the same symbol.

Every B -orbit on \mathcal{F} contains a unique T -fixed point; for $w \in W$, the Bruhat cell C_w is the B -orbit $B \cdot w$. The Schubert variety X_w is the closure of C_w ; we have $X_w = \bigcup_{x \leq w} C_x$ and so $X_x \subset X_w \iff x \leq w$.

4.1. Torus actions

Let $\rho: \mathbb{C}^* \rightarrow T$ be a cocharacter of T , and let G' be the centralizer of $T_0 = \rho(\mathbb{C}^*)$.

Theorem 8. [Sp, Theorem 6.4.7] *G' is connected and reductive; T is a maximal torus in G' . If T_0 fixes a point $B_0 \in \mathcal{F}$ so that $T_0 \subset B_0$, then $B_0 \cap G'$ is a Borel subgroup of G' .*

Let $\mathcal{F}' \cong G'/B'$ be the flag variety of G' , and put $\mathcal{F}^\rho = \mathcal{F}^{T_0}$. Using Theorem 8, we can define a G' -equivariant algebraic map $\psi: \mathcal{F}^\rho \rightarrow \mathcal{F}'$ by $\psi(B_0) = (B_0) \cap G'$.

Fix a maximal torus and Borel subgroup of G' by setting $B' = B \cap G'$, $T' = T$. The Weyl group of G' is $W' = N_{G'}(T')/T' = W \cap (G'/B')$. The Schubert varieties of \mathcal{F}' defined by the action of B' are indexed by elements of W' ; denote them by $X'_w, w \in W'$.

Proposition 9. *W' is a parabolic subgroup of W , and all parabolic subgroups arise in this way for some choice of ρ .*

This is well known; the groups G' which arise this way are Levi subgroups of parabolic subgroups of G . The second half of the statement (which is the only part we need) can be deduced from [Sp, 6.4.3 and 8.4.1], for instance.

Now we can connect the pattern map ϕ defined by Theorem 2 to geometry.

Theorem 10. *The map ψ restricts to an isomorphism on each connected component of \mathcal{F}^ρ . The restriction $\psi|_{\mathcal{F}^T}: \mathcal{F}^T \rightarrow (\mathcal{F}')^T$ is the pattern map ϕ , using the identifications $\mathcal{F}^T = W, (\mathcal{F}')^T = W'$. In particular, the components of \mathcal{F}^ρ are in bijection with $W' \setminus W$.*

Proof. To show the first assertion, it is enough to show that ψ is a finite map, since it is G' -equivariant and its image \mathcal{F}' is maximal among the compact homogeneous spaces for G' . But $\psi(g \cdot B) \in (\mathcal{F}')^T \implies T \subset g \cdot B \implies g \cdot B \in \mathcal{F}^T$, a finite set.

Certainly ψ takes T -fixed points to T -fixed points, so it induces a function $W \rightarrow W'$ by restriction. We need to show that it satisfies the properties of Theorem 2. The W' -equivariance (a) follows immediately from the G' -equivariance of ψ .

To see property (b), take $x \in W$ and $w \in W'$, and suppose that $\psi(x) \leq' \psi(wx)$. This implies that $\psi(x) \in \overline{B' \cdot \psi(wx)}$, and since x and wx lie in the same component of \mathcal{F}^ρ , we must have $x \in \overline{B' \cdot wx} \subset \overline{B \cdot wx}$. Thus $x \leq wx$. \square

Remark. Given $w \in W$, let $Y \cong \mathcal{F}'$ be the component of \mathcal{F}^ρ which contains w . Then one can show that $X_w \cap Y \cong X'_{\phi(w)}$. Therefore, $X'_{\phi(w)}$ singular implies that X_w is singular, using the result of Fogarty and Norman [FN]: a linearly algebraic group G is linearly reductive (this class includes all tori) if and only if for all smooth algebraic G -schemes X the fixed point scheme X^G is smooth.

4.2. Hyperbolic localization

Let X be a normal complex variety with an action of \mathbb{C}^* . Let $X^\circ = X^{\mathbb{C}^*}$, and let $X_1^\circ \dots X_r^\circ$ be the connected components of X° . For $1 \leq k \leq r$, define a variety

$$X_k^+ = \{x \in X \mid \lim_{t \rightarrow 0} t \cdot x \in X_k^\circ\},$$

and let X^+ be the disjoint (disconnected) union of all the X_k^+ . The inclusions $X_k^\circ \subset X_k^+ \subset X$ induce maps

$$X^\circ \xrightarrow{f} X^+ \xrightarrow{g} X.$$

Let $D^b(X)$ denote the constructible derived category of \mathbb{Q} -sheaves on X .

Definition. Given $S \in D^b(X)$, define its *hyperbolic localization*

$$S^{!*} = f^! g^* S \in D^b(X^\circ).$$

Hyperbolic localization is better adapted to \mathbb{C}^* -equivariant geometry than ordinary restriction. It was first studied by Kirwan [Ki], who showed that if S is the intersection cohomology sheaf of a projective variety with a linear \mathbb{C}^* -action, then S and $S^{!*}$ have isomorphic hypercohomology groups.

We will need two properties of hyperbolic localization from [Br]. For any $S \in D^b(X)$ and $p \in X$, we let $\chi_p(X)$ denote the Euler characteristic of the stalk cohomology at p .

Proposition 11. [Br, Proposition 3] *If $p \in X^\circ$, then*

$$\chi_p(S) = \chi_p(S^{!*}).$$

Further, hyperbolic localization satisfies a decomposition theorem [Br, Theorem 2]. When applied to $X = \mathcal{F}$ and the action given by ρ , this gives the following.

Theorem 12. *Let L_w , and L'_v be the intersection cohomology sheaves of the Schubert varieties X_w and X'_v , respectively. For any $w \in W$ and $1 \leq k \leq r$, there is an isomorphism*

$$\psi_*((L_w)^{!*}|_{\mathcal{F}_k^\circ}) \cong \bigoplus_{j=1}^m L'_{v_j}[d_j],$$

for some $v_j \in W'$ (not necessarily distinct) and $d_j \in 2\mathbb{Z}$.

Here we use the fact that hyperbolic localization preserves B' -equivariance. The fact that $d_j \in 2\mathbb{Z}$ follows from the purity of the stalks of simple mixed Hodge modules of Schubert varieties.

4.3. Proof of Theorem 4

The description of Kazhdan–Lusztig polynomials as the local intersection cohomology Poincaré polynomials of Schubert varieties [KL2] implies that for any $u, v \in W$, we have

$$P_{u,v}(1) = \chi_u(L_v) = \sum_i \dim_{\mathbb{Q}} \mathbb{H}^{2i}((L_v)_u).$$

Now, given $x, w \in W$, let \mathcal{F}_k° be the component of \mathcal{F}^ρ which contains x , and thus all of $W'x$. For every $y \in W'$, let a_y be the number of j for which $v_j = y$ in Theorem 12.

For any $z \in W'x$ we have, using Theorem 12 and Proposition 11,

$$\begin{aligned} P_{z,w}(1) &= \chi_z(L_w) = \chi_{\phi(z)}(\psi_*((L_w)^{!*}|_{\mathcal{F}_k^\circ})) \\ &= \sum_{j=1}^m \chi_{\phi(z)}(L'_{v_j}[d_j]) \\ &= \sum_{y \in W'z} a_y P'_{\phi(z), \phi(y)}(1) \end{aligned} \tag{2}$$

(note that the shift $[d_j]$ does not change the Euler characteristic, since $d_j \in 2\mathbb{Z}$).

If $z \notin [1, w]$, then equation (2) implies $a_z = 0$, since $P_{z,w} = 0$, $P'_{z,z} = 1$, and all the terms in the sum are nonnegative. Using (2) again shows that if $y \in M(x; w; W')$, i.e., y is maximal in $[1, w] \cap W'x$, then $a_y = P_{y,w}(1)$. Finally, evaluating (2) at x and keeping only the terms with $y \in M(x, w; W')$ proves Theorem 4. \square

4.4. Proof of Theorem 5

Suppose first that $x^{-1}W'x = W_I$ is a standard parabolic subgroup. Take μ to be any dominant integral cocharacter which annihilates a root α_r if and only if $r \in W'$, and let $\rho = Ad(x)\mu$. Then the action of ρ is *completely repelling* near the component \mathcal{F}_k° of \mathcal{F}^ρ which contains $W'x = xW_I$, meaning that $\mathcal{F}_k^+ = \mathcal{F}_k^\circ$, in the notation of §4.2.

This implies that hyperbolic localization to \mathcal{F}_k° is just ordinary restriction: setting $h: \mathcal{F}_k^\circ \rightarrow \mathcal{F}^\rho$ for the inclusion, we have

$$(S^{!*})|_{\mathcal{F}_k^\circ} = h^! f^! g^* S = (fh)^! g^* S = (fh)^* g^* S = S|_{\mathcal{F}_k^\circ},$$

since both h and fh are open immersions. The same argument given for Theorem 4 now proves Theorem 5, using local Poincaré polynomials instead of local Euler characteristics.

If instead $W' = W_I$, we can use the anti-involution $g \mapsto g^{-1}$ to replace left cosets by right cosets, since $P_{x^{-1}, w^{-1}} = P_{x, w}$ for all $x, w \in W$.

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