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ON METEORS, EARTHWORMS AND WIMPS

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We study a model of mass redistribution on a finite graph. We address the questions of convergence to equilibrium and the rate of convergence. We present theorems on the distribution of empty sites and the distribution of mass at a fixed vertex. These distributions are related to random permutations with certain peak sets.

1. Introduction. We study a model of mass redistribution on a finite graph. A vertex x of the graph holds mass $M_t^x \geq 0$ at time t. When a "meteor hits" x at time t, the mass M_t^x of the soil present at x is distributed equally among all neighbors of x (added to their masses). There is no soil (mass) left at x just after a meteor hit. Meteor hits are modeled as independent Poisson processes, one for each vertex of the graph.

We will address the following questions about the meteor model. Does the process converge to equilibrium? If so, at what rate? Assuming that the mass distribution process is in equilibrium, what is the distribution of "meteor craters" (sites with zero mass) at a fixed time? In equilibrium, at a fixed time and vertex, what is the distribution of soil mass? We will answer some of these questions in the asymptotic sense, for some families of growing graphs.

We will also study an "earthworm model" in which the soil redistribution events do not occur according to the Poisson arrival process but along the trajectory of a symmetric random walk on the graph. See Section 7 for the motivation of the "earthworm" model.

We will now present sources of inspiration, motivation and possible applications for our main model.

(i) Similar models of mass redistribution appeared in [24], but that paper went in a completely different direction. It was mostly focused on the limit model when the graph approximates the real line. Continuous mass redistribution also appeared in a version of the chip-firing model in [9], but the updating mechanism in that paper is different from ours. Mass redistribution is a part of every sandpile model,

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including a "continuous" version studied in [20]. Sandpile models have considerably different structures and associated questions from ours. Our model is one of the simplest models for mass redistribution. Therefore, its analysis is likely to be most complete on the mathematical side—a program that we only start in this paper. The elementary character of our model makes it amenable to a variety of mathematical techniques—something that we demonstrate in this article. Our model can be easily modified and generalized to accommodate the needs of applied science.

(ii) More on the theoretical side, our model is related to products of random matrices. Let G be a finite graph with a set of vertices $V = \{1, 2, \dots, n\}$. Let A denote the transpose of the transition probability matrix of the nearest neighbor simple random walk on this graph, and let I be the diagonal (identity) matrix. For every $i = 1, 2, \dots, n$, let $\hat{A}_i$ be the matrix obtained from $A - I$ by zeroing out all but the ith column, and let $A_i = I + \hat{A}_i$. Consider this collection of matrices $\{A_1, \dots, A_n\}$. Suppose we generate a sequence of i.i.d. random variables $\{I_1, I_2, \dots\}$ with the uniform distribution in V and consider the sequence of products of i.i.d. matrices $A_{I_n} A_{I_{n-1}} \cdots A_{I_2} A_{I_1}$. It is easy to see that M_t^x is the xth coordinate of the product of a finite sequence of matrices A_{I_j} right-multiplied with the column vector M_0^x. Does the product $A_{I_n} A_{I_{n-1}} \cdots A_{I_2} A_{I_1}$ have a limit in some sense as n tends to infinity? Products of i.i.d. random matrices have been an old and fascinating subject (see [16, 21]), and several conditions are known for convergence of distributions of the products. There are also a number of theorems on the limit distribution. However, the particular class of products considered here is just beyond the assumptions under which general results are known to hold. Most of the entries in any A_i are zero, violating assumptions in [21], equation (3.1), and these matrices are not strong contractions in the sense of [16], Theorem 1.1. However, as we will show, the additional graph structure in the background determines the limit and the rate of convergence of the products $A_{I_n} A_{I_{n-1}} \cdots A_{I_2} A_{I_1}$.

(iii) A more recent line of investigation related to our work is on Markov chains on the space of partitions; see [13, 14]. One of the important considerations in this regard is the product of i.i.d. picks from a probability measure on the space of finite probability transition matrices. That is, $S_1, S_2, \dots$ are i.i.d. stochastic matrices, and one is interested in the *backward* product $S_1 S_2 \cdots S_m$. The knowledge of this limit determines the behavior of a corresponding Markov chain on the space of finite partitions of $\mathbb{N}$; see [13], equation (5) and Theorem 1.2. The transpose of each A_i is a stochastic matrix. If we define $S_i = A_i'$, then $S_1 S_2 \cdots S_m$ is the transpose of $A_m \cdots A_2 A_1$. Hence, the limits described in this work give explicit information about certain Markov chains on the space of partitions.

In the title of this paper, WIMPs stands for "weakly interacting mathematical particles." It turns out that one of the main technical tools in this paper is a pair of "weakly interacting" continuous time symmetric random walks on the graph. If the two random walks are at different vertices, they move independently. However, if they are at the same vertex, their next jumps occur at the same time, after an exponential waiting time, common to both processes. The dependence ends here—the two processes jump to vertices chosen independently, even though they jump at the same time. Heuristically, one expects WIMPs to behave very much like independent continuous time random walks. Showing this is the heart of a number of arguments but it proves to be harder than one would expect. In other cases, the slight dependence manifests itself clearly and generates phenomena that otherwise would be trivial. WIMPs played an important role in [19].

The rest of the paper is organized as follows. Section 2 contains rigorous definitions of the meteor process and WIMPs. Section 3 is devoted to basic properties of the meteor process and convergence to equilibrium. We present three theorems on convergence. The first one is very abstract and does not provide any useful information on the rate of convergence. The second one provides a rate of convergence, but since it applies to all meteor processes, it is not optimal in specific examples. The third theorem is limited to tori and gives a sharp estimate for the convergence rate. Section 4 is devoted to the distribution of craters in circular graphs and is the most combinatorial of all the sections—it is partly based on results from [3]. We address several questions about craters. The first one is concerned with the probability of a given pattern of craters. The second question is about fluctuations of the numbers of patterns around the mean. We do not provide a standard large deviations result, but we prove a theorem on the most likely configuration of craters assuming that there are very few of them. Section 5 presents results on the mass distribution at a single vertex or a family of vertices, in case of circular graphs. Section 6 contains theorems on the mass distribution for noncircular graphs. The first result is a bound for the variance for a large class of graphs. The second result is a completely explicit limiting distribution at a vertex, for the complete graphs, when the size of the graph grows to infinity. Finally, Section 7 contains the proof of the claim that the earthworm distributes mass in a torus more or less evenly, on a large scale.

2. Preliminaries.

The following setup and notation will be used in most of the paper. All constants will be assumed to be strictly positive, finite, real numbers, unless stated otherwise. The notation $|S|$ will be used for the cardinality of a finite set, S.

We will consider only finite connected graphs with no loops and no multiple edges. We will often denote the chosen graph by G and its vertex set by V. In particular, we often use k for $|V|$. We let d_v stand for the degree of a vertex v, and write $v \leftrightarrow x$ if vertices v and x are connected by an edge.

We will write C_k to denote the circular graph with k vertices, $k \geq 2$. In other words, the vertex set of C_k is $\{1, 2, \dots, k\}$, and the only pairs of vertices joined by edges are of the form $(j, j+1)$ for $j = 1, 2, \dots, k-1$, and $(k, 1)$. For C_k, all arguments will apply "mod k." For example, we will refer to k as a vertex "to the left of 1," and interpret $j - 1$ as k in the case when $j = 1$.

Every vertex v is associated with a Poisson process N^v representing "arrival times of meteors" with intensity 1. We assume that processes N^v are jointly independent. A vertex v holds some "soil" with mass equal to $M_t^v \geq 0$ at time $t \geq 0$. The processes M^v evolve according to the following scheme.

We assume that $M_0^v \in [0, \infty)$ for every v, a.s. At the time t of a jump of N^v, M^v jumps to 0. At the same time, the mass M_{t-}^v is "distributed" equally among all adjacent sites, that is, for every vertex $x \leftrightarrow v$, the process M^x increases by M_{t-}^v/d_v; more formally, $M_t^x = M_{t-}^x + M_{t-}^v/d_v$. The mass M^v will change only when N^v jumps and just prior to that time there is positive mass at v, or N^x jumps for some $x \leftrightarrow v$ and just prior to that time there is positive mass at x. We will denote the mass process $\mathcal{M}_t = \{M_t^v, v \in V\}$.

We will now define WIMPs ("weakly interacting mathematical particles") which will be used in a number of arguments.

DEFINITION 2.1. We will define several processes on the same probability space. Suppose that two mass processes $\mathcal{M}_0$ and $\widetilde{\mathcal{M}}_0$ are given, and assume that $a = \sum_{v \in V} M_0^v = \sum_{v \in V} \widetilde{M}_0^v$.

For each $j \geq 1$, let $\{Y_n^j, n \geq 0\}$ be a discrete time symmetric random walk on G with the initial distribution $\mathbb{P}(Y_0^j = x) = M_0^x/a$ for $x \in V$. Similarly, let $\{\widetilde{Y}_n^j, n \geq 0\}$, $j \geq 1$, be discrete time symmetric random walks on G with the initial distribution $\mathbb{P}(\widetilde{Y}_0^j = x) = \widetilde{M}_0^x/a$ for $x \in V$, $j \geq 1$. We assume that conditional on $\mathcal{M}_0$ and $\widetilde{\mathcal{M}}_0$, all processes $\{Y_n^j, n \geq 0\}$, $j \geq 1$ and $\{\widetilde{Y}_n^j, n \geq 0\}$, $j \geq 1$, are independent.

Recall Poisson processes N^v defined earlier in this section, and assume that they are independent of $\{Y_n^j, n \geq 0\}$, $j \geq 1$ and $\{\widetilde{Y}_n^j, n \geq 0\}$, $j \geq 1$. For every $j \geq 1$, we define a continuous time Markov process $\{Z_t^j, t \geq 0\}$ by requiring that the embedded discrete time Markov chain for Z^j is Y^j and Z^j jumps at a time t if and only if N^v jumps at time t, where $v = Z_{t-}^j$. We define $\{\widetilde{Z}_t^j, t \geq 0\}$ in an analogous way, so that the embedded discrete time Markov chain for $\widetilde{Z}^j$ is $\widetilde{Y}^j$ and $\widetilde{Z}^j$ jumps at a time t if and only if N^v jumps at time t, where $v = \widetilde{Z}_{t-}^j$. Note that the jump times of all Z^j's and $\widetilde{Z}^j$'s are defined by the same family of Poisson processes $\{N^v\}_{v \in V}$.

REMARK 2.2. The processes Z^j and $\widetilde{Z}^j$ in Definition 2.1 are continuous time nearest neighbor symmetric random walks on G with exponential holding time with mean 1.

The joint distribution of (Z^1, Z^2) is the following. The state space for the process (Z^1, Z^2) is V^2. If $(Z_t^1, Z_t^2) = (x, y)$ with $x \neq y$, then the process will stay in this state for an exponential amount of time with mean $1/2$, and at the end of this time interval, one of the two processes (chosen uniformly) will jump to one of the nearest neighbors (also chosen uniformly). This behavior is the same as that of two independent random walks. However, if $(Z_t^1, Z_t^2) = (x, x)$, then the pair of processes behave in a way that is different from that of a pair of independent random walks. Namely, after an exponential waiting time with mean 1 (not 1/2), both processes will jump at the same time; each one will jump to one of the nearest neighbors of x chosen uniformly and independently of the direction of the jump of the other process.

The same remark applies to any pair of processes in the family $\{Z^j, j \geq 1\} \cup \{\widetilde{Z}^j, j \geq 1\}$.

3. Basic properties and convergence to equilibrium. It will be convenient from the technical point of view to postpone the presentation of the most elementary properties of the meteor process to the end of this section. We will start with three theorems on convergence to the stationary distribution.

REMARK 3.1. The mass process $\{\mathcal{M}_t, t \geq 0\}$ is a somewhat unusual stochastic process in that its state space can be split into an uncountable number of disjoint communicating classes. It is easy to see that, due to the definition of the evolution of $\{\mathcal{M}_t, t \geq 0\}$, for every time $t \geq 0$ and every $v \in V$, $M_t^v = \sum_{x \in V} a_x M_0^x$, where a_x is a random variable that depends on t and v. Every a_x has the form $m \prod_{y \in V} d_y^{-j_y}$ for some integers m and j_y. In other words, a_x take values in the ring $\mathbb{Z}[1/e_1, \dots, 1/e_i]$, where $e_1, \dots, e_i$ is a list of all distinct values of degrees of vertices in V. Therefore, M_t^v's take values in the free module over the ring $\mathbb{Z}[1/e_1, \dots, 1/e_i]$ spanned by $\{M_0^v, v \in V\}$.

For example, consider the following two initial distributions. Suppose that $M_0^v = 1$ for all v. Fix some $x \in V$, and let $\widetilde{M}_0^v = 1/\pi$ for all $v \neq x$ and $\widetilde{M}_0^x = |V| - (|V| - 1)/\pi$. If $\{\mathcal{M}_t, t \geq 0\}$ and $\{\widetilde{\mathcal{M}}_t, t \geq 0\}$ are mass processes with these initial distributions, then for every $t > 0$, the distributions of $\mathcal{M}_t$ and $\widetilde{\mathcal{M}}_t$ will be mutually singular.

It follows from these observations that proving convergence of $\{\mathcal{M}_t, t \geq 0\}$ to the stationary distribution cannot proceed along the most classical lines; see [22] for the discussion of this technical issue and a solution. We will follow [22] in spirit although not in all technical details.

THEOREM 3.2. *Consider the process $\{\mathcal{M}_t, t \geq 0\}$ on a graph G. Assume that $|V| = k$ and $\sum_{v \in V} M_0^v = k$. When $t \to \infty$, the distribution of $\mathcal{M}_t$ converges to a distribution Q on $[0, k]^k$. The distribution Q is the unique stationary distribution for the process $\{\mathcal{M}_t, t \geq 0\}$. In particular, Q is independent of the initial distribution of $\mathcal{M}$.*

PROOF. We will consider a coupling of two copies of the process $\{\mathcal{M}_t, t \geq 0\}$. Suppose that $\{\mathcal{M}_t, t \geq 0\}$ and $\{\widetilde{\mathcal{M}}_t, t \geq 0\}$ are driven by the same processes

$\{N_t^v, t \geq 0\}_{v \in V}$ but the distribution of $\mathcal{M}_0$ is not necessarily the same as that $\widetilde{\mathcal{M}}_0$. We do assume that $\sum_{v \in V} M_0^v = \sum_{v \in V} \widetilde{M}_0^v = k$.

First we will argue that the total variation distance between the distributions of $\mathcal{M}_t$ and $\widetilde{\mathcal{M}}_t$, that is, $D_t := \sum_{v \in V} |M_t^v - \widetilde{M}_t^v|$ is a nonincreasing process, a.s. Note that since G is finite, the number of jumps of N^v's is finite on every finite time interval and D_t is constant between any two jump times. Suppose that N^x has a jump at a time T. Then

$$\begin{aligned}
D_T - D_{T-} &= \sum_{v \in V} |M_T^v - \widetilde{M}_T^v| - \sum_{v \in V} |M_{T-}^v - \widetilde{M}_{T-}^v| \\
&= -|M_{T-}^x - \widetilde{M}_{T-}^x| + \sum_{v \leftrightarrow x} (|M_T^v - \widetilde{M}_T^v| - |M_{T-}^v - \widetilde{M}_{T-}^v|) \\
&= -|M_{T-}^x - \widetilde{M}_{T-}^x| \\
&\quad + \sum_{v \leftrightarrow x} \left(\left| (M_{T-}^v - \widetilde{M}_{T-}^v) + \frac{M_{T-}^x - \widetilde{M}_{T-}^x}{d_x} \right| - |M_{T-}^v - \widetilde{M}_{T-}^v| \right) \\
&\leq -|M_{T-}^x - \widetilde{M}_{T-}^x| \\
&\quad + \sum_{v \leftrightarrow x} \left(|M_{T-}^v - \widetilde{M}_{T-}^v| + \left| \frac{M_{T-}^x - \widetilde{M}_{T-}^x}{d_x} \right| - |M_{T-}^v - \widetilde{M}_{T-}^v| \right) \\
&= -|M_{T-}^x - \widetilde{M}_{T-}^x| + \sum_{v \leftrightarrow x} \left| \frac{M_{T-}^x - \widetilde{M}_{T-}^x}{d_x} \right| = 0.
\end{aligned}$$

This shows that D_t is nonincreasing.

Recall that G is connected, and fix some vertex y. Let $(x_1, x_2, \dots, x_n)$ be a sequence of vertices of G such that $x_j \leftrightarrow x_{j+1}$ for $1 \leq j \leq n-1$, $x_n \leftrightarrow y$ and $\{x_1, x_2, \dots, x_n\} = G \setminus \{y\}$. The vertices' x_j's are not necessarily distinct. Recall that $|V| = k$. Let $d = \max_{x \in V} d_x$. Let $a = \max_{v \neq y} M_0^v$, $b = \sum_{v \neq y} M_0^v$, and note that $a \geq b/(k-1)$. Suppose that the first n meteors hit vertices $x_1, x_2, \dots, x_n$, in this order. During this process, at least $1/d$th part of the mass from any vertex x_j, $j < n$, is pushed to x_{j+1}, and at least $1/d$th part of the mass at x_n is pushed to y. Let m be the smallest integer with the property that $M_0^{x_m} = a$. Then at least ad^{-1} of mass will be pushed from x_m to x_{m+1}. This implies that least ad^{-2} of mass will be pushed from x_{m+1} to x_{m+2}. By induction, at least ad^{-j} of mass will be pushed from x_{m+j-1} to x_{m+j}. Hence, at least ad^{-n} of mass will be added to y. In other words, the mass outside y will be reduced at least by $ad^{-n} \geq bd^{-n}/(k-1)$. Putting it in a different way, the mass outside y will be reduced at least by the factor of $1 - d^{-n}/(k-1)$.

Consider an arbitrarily small $\varepsilon > 0$, and let m be so large that $(1 - d^{-n}/(k-1))^m k \leq \varepsilon$. If the first nm meteors hit vertices

$$\underbrace{x_1, x_2, \dots, x_n, \qquad x_1, x_2, \dots, x_n, \qquad \dots, \qquad x_1, x_2, \dots, x_n,}_{m \text{ times}}$$

in this order, then the mass outside y will be reduced to at most $(1 - d^{-n}/(k - 1))^m k \le \varepsilon$. Sooner or later, with probability 1, there will be a sequence of nm meteor hits described above, and then the mass outside y will be less than ε. Hence the mass at y will be between $k - \varepsilon$ and k at the end of this sequence of meteor hits. Note that the argument applies equally to $\{\mathcal{M}_t, t \ge 0\}$ and $\{\widetilde{\mathcal{M}}_t, t \ge 0\}$. Hence, at the end of this sequence of nm hits, the function D will be at most 2ε. Since D_t is nonincreasing, we see that D_t converges to 0, a.s.

For every t, the distribution of $\mathcal{M}_t$ is a measure on $[0, k]^k$, a compact set, so the family of distributions of $\mathcal{M}_t$, $t \ge 0$, is tight. Therefore, there exists a sequence t_n converging to ∞ such that the distributions of $\mathcal{M}_{t_n}$ converge to a distribution Q on $[0, k]^k$, as $n \to \infty$.

Let $\mathbf{d}$ denote the Prokhorov distance (see [4], page 238) between probability measures on $[0, k]^k$, and recall that convergence in the metric $\mathbf{d}$ is equivalent to the weak convergence of measures. By abuse of notation, we will use the same symbol for the Prokhorov distance between probability measures on $[0, k]^k$ and $\mathbb{R}$. We will also apply $\mathbf{d}$ to random variables, with the understanding that it applies to their distributions. Let $\mathbf{0}$ denote the probability distribution on $\mathbb{R}$ concentrated at 0. It is easy to see that for every $\delta > 0$ there exists $\alpha(\delta) > 0$ such that if $\mathbf{d}(D_t, \mathbf{0}) \le \alpha(\delta)$ then $\mathbf{d}(\mathcal{M}_t, \widetilde{\mathcal{M}}_t) \le \delta$.

The bounds in our argument showing convergence of D_t to 0 do not depend on $\mathcal{M}_0$ or $\widetilde{\mathcal{M}}_0$ so there exists a deterministic function $\rho : [0, \infty) \to [0, \infty)$ such that $\lim_{t\to\infty} \rho(t) = 0$ and $\mathbf{d}(D_t, \mathbf{0}) \le \rho(t)$ for any $\mathcal{M}_0$ any $\widetilde{\mathcal{M}}_0$.

Suppose that there exists a sequence s_n converging to ∞ such that the distributions of $\mathcal{M}_{s_n}$ converge to a distribution Q' on $[0, k]^k$, as $n \to \infty$, and $Q' \ne Q$. Let $\delta = \mathbf{d}(Q, Q')/2 > 0$.

Find u_0 so large that $\rho(t) < \alpha(\delta)$ for $t \ge u_0$. Let t_n and s_m be such that $u_0 < t_n < s_m$, $\mathbf{d}(\mathcal{M}_{t_n}, Q) \le \delta/4$ and $\mathbf{d}(\mathcal{M}_{s_m}, Q') \le \delta/4$. Let $\widetilde{\mathcal{M}}_0 = \mathcal{M}_{s_m - t_n}$. Then $\mathbf{d}(\widetilde{\mathcal{M}}_{t_n}, Q') \le \delta/4$. Since $t_n > u_0$, we have $\rho(t_n) < \alpha(\delta)$, so $\mathbf{d}(D_{t_n}, \mathbf{0}) \le \alpha(\delta)$ and, therefore, $\mathbf{d}(\mathcal{M}_{t_n}, \widetilde{\mathcal{M}}_{t_n}) \le \delta$. By the triangle inequality,

$$\begin{aligned} \mathbf{d}(Q, Q') &\le \mathbf{d}(\mathcal{M}_{t_n}, Q) + \mathbf{d}(\widetilde{\mathcal{M}}_{t_n}, Q') + \mathbf{d}(\mathcal{M}_{t_n}, \widetilde{\mathcal{M}}_{t_n}) \\ &\le \delta/4 + \delta/4 + \delta = 3\delta/2. \end{aligned}$$

This contradicts the fact that $\mathbf{d}(Q, Q') = 2\delta$ and shows that $\mathcal{M}_t$ converges in distribution to Q, as $t \to \infty$. The fact that D_t converges to 0 shows that Q does not depend on the distribution of $\mathcal{M}_0$.

Next we will show that the distribution Q is stationary. It is routine to show that for every $\eta > 0$ there exists $\beta(\eta) > 0$ such that for any distributions Q and Q' on $[0, k]^k$ with $\mathbf{d}(Q', Q'') \le \beta(\eta)$, one can construct $\mathcal{M}_0$ and $\widetilde{\mathcal{M}}_0$ on the same probability space so that the distribution of $\mathcal{M}_0$ is Q', the distribution of $\widetilde{\mathcal{M}}_0$ is Q'', and $\mathbf{d}(D_0, \mathbf{0}) \le \eta$.

Consider an arbitrarily small $\delta > 0$. Let the distribution of $\mathcal{M}_0$ be Q and find u_1 so large that $\mathbf{d}(\mathcal{M}_t, Q) \le \beta(\alpha(\delta/2)) \wedge \delta/2$ for all $t \ge u_1$. Then we can construct

$\mathcal{M}_0$ and $\widetilde{\mathcal{M}}_0$ on the same probability space so that the distribution of $\mathcal{M}_0$ is Q, the distribution of $\widetilde{\mathcal{M}}_0$ is the same as that of $\mathcal{M}_{u_1}$, and $\mathbf{d}(D_0, \mathbf{0}) \le \alpha(\delta/2)$. Then $\mathbf{d}(D_t, \mathbf{0}) \le \alpha(\delta/2)$ for all $t \ge 0$ and, therefore, we have $\mathbf{d}(\mathcal{M}_t, \widetilde{\mathcal{M}}_t) \le \delta/2$ for any $t \ge 0$. Note that $\mathbf{d}(\widetilde{\mathcal{M}}_t, Q) \le \delta/2$ for all $t \ge 0$ because $\mathbf{d}(\mathcal{M}_t, Q) \le \delta/2$ for all $t \ge u_1$. We obtain for $t \ge 0$,

$$\mathbf{d}(\mathcal{M}_t, Q) \le \mathbf{d}(\widetilde{\mathcal{M}}_t, Q) + \mathbf{d}(\mathcal{M}_t, \widetilde{\mathcal{M}}_t) \le \delta/2 + \delta/2 = \delta.$$

Since $\delta > 0$ is arbitrarily small, Q is stationary. $\square$

REMARK 3.3. In view of Theorem 3.2 and its proof, it is easy to see that there exists a stationary version of the process $\mathcal{M}_t$ on the whole real line; that is, there exists a process $\{\mathcal{M}_t, t \in \mathbb{R}\}$, such that the distribution of $\mathcal{M}_t$ is the stationary measure Q for each $t \in \mathbb{R}$. Moreover, one can construct independent Poisson processes $\{N_t^v, t \in \mathbb{R}\}$, $v \in V$, on the same probability space, such that $\{\mathcal{M}_t, t \in \mathbb{R}\}$ jumps according to the algorithm described in Section 2, relative to these Poisson processes. We set $N_0^v = 0$ for all v for definiteness.

The next theorem is the only result in our paper that is proved in a context more general than that in Section 2. Consider a graph, and let $\mathbf{P} = (p_{xy})_{x,y \in V}$ be the probability transition matrix for a Markov chain on V. In this model, if a meteor hits site x, then the mass is distributed to other vertices in proportion to p_{xy}, not necessarily in equal proportions to all neighbors. We remark parenthetically that, by convention, we place an edge between two vertices x and y of G if and only if $p_{xy} + p_{yx} > 0$.

THEOREM 3.4. *Consider a graph G, and suppose that U_t and $\widetilde{U}_t$ are independent continuous time Markov chains with mean* 1 *exponential holding times at every vertex and the transition rates for the embedded discrete time Markov chains given by* $\mathbf{P}$. *Let*

$$\begin{aligned} \tau_U &= \inf\{t \ge 0 : U_t = \widetilde{U}_t\}, \\ \alpha(t) &= \sup_{x,y \in V} \mathbb{P}(\tau_U > t \mid U_0 = x, \widetilde{U}_0 = y). \end{aligned} \tag{3.1}$$

Consider any (possibly random) distributions of mass $\mathcal{M}_0$ and $\widetilde{\mathcal{M}}_0$; that is, assume that $M_0^x \ge 0$ and $\widetilde{M}_0^x \ge 0$ for all $x \in V$ and $\sum_{x \in V} M_0^x = \sum_{x \in V} \widetilde{M}_0^x = |V|$, a.s. One can define mass processes $\mathcal{M}_t$ and $\widetilde{\mathcal{M}}_t$ on a common probability space so that for all $t \ge 0$,

$$\mathbb{E}\left(\sum_{x \in V} |M_t^x - \widetilde{M}_t^x|\right) \le |V| \alpha(t). \tag{3.2}$$

REMARK 3.5. In the notation of Theorem 3.4, let $T_x = \inf\{t \geq 0 : U_t = x\}$. According to [2], Proposition 1, if $\mathbf{P}$ represents a reversible Markov chain, then

$$\sup_{x,y\in V} \mathbb{E}(\tau_U | U_0 = x, \widetilde{U}_0 = y) \leq c \sup_{x,y\in V} \mathbb{E}(T_x | U_0 = y).$$

For an arbitrary Markov chain, Conjecture 1 in [2] states that

$$\sup_{x,y\in V} \mathbb{E}(\tau_U | U_0 = x, \widetilde{U}_0 = y) \leq c|V| \sup_{x,y\in V} \mathbb{E}(T_x | U_0 = y).$$

The conjecture remains open at this time, as far as we know.

PROOF OF THEOREM 3.4. Suppose that Z and $\widetilde{Z}$ are constructed as Z^1 and $\widetilde{Z}^1$ in Definition 2.1, except that $\{Y_n^1, n \geq 0\}$ and $\{\widetilde{Y}_n^1, n \geq 0\}$ are discrete time Markov chains with the transition probability matrix $\mathbf{P}$. The initial distributions are given by $\mathbb{P}(Y_0 = x) = M_0^x/|V|$ and $\mathbb{P}(\widetilde{Y}_0 = x) = \widetilde{M}_0^x/|V|$ for $x \in V$. Let

$$\tau = \inf\{t \geq 0 : Z_t = \widetilde{Z}_t\},$$
$$\widehat{Z}_t = \begin{cases} \widetilde{Z}_t, & \text{for } t \leq \tau, \\ Z_t, & \text{for } t > \tau. \end{cases}$$

The distribution of $\{\widehat{Z}_t, t \geq 0\}$ is the same as that of $\{\widetilde{Z}_t, t \geq 0\}$.

Let $\{Z_t^*, t \geq 0\}$ have the same distribution as $\{\widetilde{Z}_t, t \geq 0\}$ and be independent of $\{Z_t, t \geq 0\}$, given $\mathcal{M}_0$ and $\widetilde{\mathcal{M}}_0$. Let $\tau_* = \inf\{t \geq 0 : Z_t = Z_t^*\}$. Since the Poisson processes N^m are independent from one another, it follows easily that the distributions of

$$\{\tau, \{Z_t, t \in [0, \tau]\}, \{\widetilde{Z}_t, t \in [0, \tau]\}\}$$

and

$$\{\tau_*, \{Z_t, t \in [0, \tau_*]\}, \{Z_t^*, t \in [0, \tau_*]\}\}$$

are identical. Thus τ and τ_* have the same distributions, and therefore, (3.1) implies that $\mathbb{P}(\tau > t) \leq \alpha(t)$.

Let $\mathcal{G}_t = \sigma(\mathcal{M}_s, \widetilde{\mathcal{M}}_s, 0 \leq s \leq t)$, and note that $\mathcal{G}_t = \sigma(\mathcal{M}_0, \widetilde{\mathcal{M}}_0, N_s^v, 0 \leq s \leq t, v \in V)$. The process Z is "coupled" with the processes N^v which determine the motion of mass. This easily implies that for all x and t,

$$\mathbb{P}(Z_t = x | \mathcal{G}_t) = M_t^x/|V|. \tag{3.3}$$

It is easy to see that the distributions of

$$\{\{N_s^v, s \in [0, t]\}_{v\in V}, \{\widetilde{Z}_s, s \in [0, t]\}\}$$

and

$$\{\{N_s^v, s \in [0, t]\}_{v\in V}, \{\widehat{Z}_s, s \in [0, t]\}\}$$

are the same, so we obtain the following formula, analogous to (3.3),

$$\mathbb{P}(\widetilde{Z}_t = x|\mathcal{G}_t) = \mathbb{P}(\widehat{Z}_t = x|\mathcal{G}_t) = \widetilde{M}_t^x/|V|.$$

It follows that

$$\begin{aligned}
\mathbb{E}\biggl(\sum_{x\in V}|M_t^x - \widetilde{M}_t^x|\biggr) &= |V|\mathbb{E}\biggl(\sum_{x\in V}|\mathbb{P}(Z_t = x|\mathcal{G}_t) - \mathbb{P}(\widehat{Z}_t = x|\mathcal{G}_t)|\biggr)\\
&= |V|\mathbb{E}\biggl(\sum_{x\in V}|\mathbb{E}(\mathbf{1}_{\{Z_t=x\}} - \mathbf{1}_{\{\widehat{Z}_t=x\}}|\mathcal{G}_t)|\biggr)\\
&\le |V|\mathbb{E}\biggl(\sum_{x\in V}\mathbb{E}(|\mathbf{1}_{\{Z_t=x\}} - \mathbf{1}_{\{\widehat{Z}_t=x\}}||\mathcal{G}_t)\biggr)\\
&= |V|\mathbb{E}\mathbb{E}(\mathbf{1}_{\{Z_t\neq\widehat{Z}_t\}}|\mathcal{G}_t)\\
&= |V|\mathbb{E}\mathbb{P}(\tau > t|\mathcal{G}_t) \le |V|\alpha(t).
\end{aligned}$$

This completes the proof. $\square$

THEOREM 3.6. *Consider the meteor process on a graph $G = C_n^d$ (the product of d copies of the cycle C_n). Consider any distributions (possibly random) of mass $\mathcal{M}_0$ and $\widetilde{\mathcal{M}}_0$; that is, assume that $M_0^x \ge 0$ and $\widetilde{M}_0^x \ge 0$ for all $x \in V$ and $\sum_x M_0^x = \sum_x \widetilde{M}_0^x = |V| = n^d$, a.s. There exist constants c_1, c_2 and c_3, not depending on n and d, such that if $n \ge 1 \vee c_1\sqrt{d\log d}$ and $t \ge c_2dn^2$, then one can define a coupling of processes $\mathcal{M}_t$ and $\widetilde{\mathcal{M}}_t$ on a common probability space so that*

$$\mathbb{E}\biggl(\sum_{x\in V}|M_t^x - \widetilde{M}_t^x|\biggr) \le \exp\bigl(-c_3t/(dn^2)\bigr)|V|. \tag{3.4}$$

PROOF. *Step* 1. In this step, we will show that there exist constants $c_1, c_2 \in (0,\infty)$ and $c_4 < 2$, not depending on n and d, such that if $n \ge 1 \vee c_1\sqrt{d\log d}$ and $t \ge c_2dn^2$, and the processes $\mathcal{M}_t$ and $\widetilde{\mathcal{M}}_t$ are independent, then

$$\mathbb{E}\biggl(\sum_{x\in V}|M_t^x - \widetilde{M}_t^x|\biggr) \le c_4|V|. \tag{3.5}$$

Let Z and $\widetilde{Z}$ be defined as Z^1 and Z^2 in Definition 2.1. In particular, $\mathbb{P}(Z_0 = x) = \mathbb{P}(\widetilde{Z}_0 = x) = M_0^x/|V|$ for $x \in V$.

Let $Z_t^* = Z_t - \widetilde{Z}_t$, and note that Z^* is a continuous time Markov process on V, with the mean holding time equal to $1/2$ at all vertices $x \neq \mathbf{0} := (0,\dots,0)$. Recall that if $(Z_t, \widetilde{Z}_t) = (x, x)$, then after an exponential waiting time with mean 1 (not $1/2$), both processes will jump at the same time. They will jump to one of the neighbors of x (the same for both processes) with probability $1/(2d)$. Hence, this jump of $(Z, \widetilde{Z})$ will not correspond to a jump of Z^*. It follows that the mean holding time for Z^* at $\mathbf{0}$ is $\beta := (1 - 1/(2d))^{-1}$. Note that if $Z_t^* = \mathbf{0}$, the next jump it will take will be to a vertex at the distance 2 from $\mathbf{0}$. If $Z_t^* \neq \mathbf{0}$, then the next jump will be to a neighbor of Z_t^*.

Let Z_t^1 be a continuous time symmetric nearest neighbor random walk on V, with the mean holding time equal to $1/2$ at all vertices $x \neq \mathbf{0}$, and mean holding time at $\mathbf{0}$ equal to β. The only difference between Z^1 and Z^* is that Z^1 can jump from $\mathbf{0}$ only to a nearest neighbor while Z^* can jump from $\mathbf{0}$ to some other vertices.

We will construct a coupling of Z^* and Z^1 such that $Z_0^1 = Z_0^*$ and, a.s.,

$$\{t \geq 0 : Z_t^* = \mathbf{0}\} \subset \{t \geq 0 : Z_t^1 = \mathbf{0}\}. \tag{3.6}$$

We let $Z_t^1 = Z_t^*$ for all t less than the time S_1 of the first jump out of $\mathbf{0}$. At the time S_1, we let processes Z^1 and Z^* make independent jumps, each one according to its own jump distribution.

Let $T_{Z^1}(\mathbf{0}) = \inf\{t \geq 0 : Z_t^1 = \mathbf{0}\}$, and let $T_{Z^*}(\mathbf{0})$ have the analogous meaning. Suppose that $x, y \in V$, $x \leftrightarrow \mathbf{0}$ and $y \not\leftrightarrow \mathbf{0}$. Then for every $t \geq 0$,

$$\mathbb{P}^x(T_{Z^1}(\mathbf{0}) > t) \leq \mathbb{P}^y(T_{Z^*}(\mathbf{0}) > t)$$

because Z^* has to pass a neighbor of $\mathbf{0}$ on its way to $\mathbf{0}$. Now standard coupling arguments show that we can construct Z^1 after S_1 in such a way that it hits $\mathbf{0}$ at the same time or earlier than the hitting time of $\mathbf{0}$ by Z^*. Let S_2 be the first hitting time of $\mathbf{0}$ by Z^1 after time S_1. We will consider several cases:

(a) Suppose that $Z_{S_2}^* \not\leftrightarrow \mathbf{0}$. We let processes Z^1 and Z^* evolve independently after S_2 until the first time S_3 such that either $Z_{S_3}^1 \neq \mathbf{0}$ or $Z_{S_3}^* \leftrightarrow \mathbf{0}$.

(a1) Suppose that $Z_{S_3}^1 \neq \mathbf{0}$. Then $Z_{S_3}^1 \leftrightarrow \mathbf{0}$ and $Z_{S_3}^* \not\leftrightarrow \mathbf{0}$. Hence, we can couple Z^1 and Z^* after time S_3 in such a way that Z^1 will hit $\mathbf{0}$ before Z^* does. At the time when Z^1 hits $\mathbf{0}$, we are back in the case represented by the time S_2.

(a2) Suppose that $Z_{S_3}^* \leftrightarrow \mathbf{0}$. Then $Z_{S_3}^1 = \mathbf{0}$. We continue the construction of the processes after S_3 as in case (b) described below.

(b) Suppose that $Z_{S_2}^* \leftrightarrow \mathbf{0}$. We let processes Z^1 and Z^* evolve independently after S_2 until the first time S_4 such that either Z^1 or Z^* jumps.

(b1) If Z^* jumps at time S_4 and $Z_{S_4}^* \not\leftrightarrow \mathbf{0}$, then we are back in the case analogous to (a).

(b2) If Z^* jumps at time S_4 and $Z_{S_4}^* = \mathbf{0}$, then we continue in the same way as after time 0.

(b3) If Z^1 jumps at time S_4, then $Z_{S_4}^* \leftrightarrow \mathbf{0}$ and $Z_{S_4}^1 \leftrightarrow \mathbf{0}$. Then we couple Z^1 and Z^* after S_4 so that they hit $\mathbf{0}$ at the same time. We continue after this time in the same way as after time 0.

(c) Suppose that $Z_{S_2}^* = \mathbf{0}$. Then we continue after this time in the same way as after time 0.

The construction of Z^1 can be continued by induction. This completes the argument justifying the existence of a coupling of Z^1 and Z^* such that Z^1 is at $\mathbf{0}$ whenever Z^* is at this point.

It is elementary to check that for some c_3 and all $d, n \geq 2$, $j \geq c_3 d n^2$ and $x \in V$, we have

$$n^{-d}/2 \leq \mathbb{P}(Y_j \in x) \leq 2n^{-d}. \tag{3.7}$$

Let N^* be a Poisson process with the mean time between jumps equal to β. It is easy to see that there exists $c_4 > 0$ such that for $t \geq 2\beta c_3 d n^2$,

$$\mathbb{P}(N_t^* \leq c_3 d n^2) \leq e^{-c_4 n^2}. \tag{3.8}$$

Let $\widetilde{N}_t$ be the number of jumps made by Z^1 by the time t, and note that $\widetilde{N}$ is stochastically minorized by N^*. By (3.6), (3.7) and (3.8), there are c_5 and c_6 such that for $n \geq c_5 \sqrt{d \log d}$ and $t \geq 2\beta c_3 d n^2$,

$$\begin{aligned}
\mathbb{P}(Z_t^* = \mathbf{0}) &\leq \mathbb{P}(Z_t^1 = \mathbf{0}) \\
&= \sum_{j=0}^{\infty} \mathbb{P}(Z_t^1 = \mathbf{0} | \widetilde{N}_t = j) \mathbb{P}(\widetilde{N}_t = j) \\
&\leq \mathbb{P}(\widetilde{N}_t \leq c_3 d n^2) + \sum_{j > c_3 d n^2} \mathbb{P}(Z_t^1 = \mathbf{0} | \widetilde{N}_t = j) \mathbb{P}(\widetilde{N}_t = j) \\
&\leq \mathbb{P}(N_t^* \leq c_3 d n^2) + \sum_{j > c_3 d n^2} 2n^{-d} \mathbb{P}(\widetilde{N}_t = j) \\
&\leq e^{-c_4 n^2} + 2n^{-d} \\
&\leq c_6 n^{-d}.
\end{aligned} \tag{3.9}$$

From now on, we will assume that $n \geq c_5 \sqrt{d \log d}$ and $t \geq 2\beta c_3 d n^2$.

Let $\widehat{N}_t$ be the number of jumps made by Z by the time t and note $\widehat{N}$ is stochastically minorized by N^*. By (3.7)–(3.8), for $x \in V$,

$$\begin{aligned}
\mathbb{P}(Z_t = x) &= \sum_{j=0}^{\infty} \mathbb{P}(Z_t = x | \widehat{N}_t = j) \mathbb{P}(\widehat{N}_t = j) \\
&\geq \sum_{j > c_3 d n^2} \mathbb{P}(Z_t = x | \widehat{N}_t = j) \mathbb{P}(\widehat{N}_t = j) \\
&\geq \sum_{j > c_3 d n^2} (n^{-d}/2) \mathbb{P}(\widehat{N}_t = j) \\
&\geq (n^{-d}/2) \mathbb{P}(\widehat{N}_t > c_3 d n^2) \\
&\geq (n^{-d}/2) \mathbb{P}(N_t^* > c_3 d n^2) \\
&\geq c_7 n^{-d}.
\end{aligned} \tag{3.10}$$

It follows from (3.9) that $\mathbb{P}(Z_t - \widetilde{Z}_t = \mathbf{0}) = \mathbb{P}(Z_t^* = \mathbf{0}) \le c_6 n^{-d}$, so for fixed t and n, there must exist $V_1 \subset V$ with $|V_1| \ge n^d/2$, such that for all $x \in V_1$,

$$\mathbb{P}(Z_t = \widetilde{Z}_t = x) \le 2c_6 n^{-2d}. \tag{3.11}$$

Let $\mathcal{G}_t = \sigma(\mathcal{M}_s, 0 \le s \le t)$. It follows easily from the definition of Z that for $x \in V$,

$$\mathbb{P}(Z_t = x | \mathcal{G}_t) = M_t^x / n^d$$

and, by (3.10),

$$\mathbb{E} M_t^x = n^d \mathbb{E}\mathbb{P}(Z_t = x | \mathcal{G}_t) = n^d \mathbb{P}(Z_t = x) \ge c_7. \tag{3.12}$$

The random variables Z_t and $\widetilde{Z}_t$ are conditionally independent given $\mathcal{G}_t$, so for $x \in V$,

$$\mathbb{P}(Z_t = \widetilde{Z}_t = x | \mathcal{G}_t) = (M_t^x / n^d)^2.$$

Thus, by (3.11), for $x \in V_1$,

$$\mathbb{E}(M_t^x)^2 = n^{2d} \mathbb{E}\mathbb{P}(Z_t = \widetilde{Z}_t = x | \mathcal{G}_t) = n^{2d} \mathbb{P}(Z_t = \widetilde{Z}_t = x) \le 2c_6.$$

Let $c_8 = \sqrt{2c_6}$. We have for $j \ge 1$, $x \in V_1$,

$$\mathbb{P}(2^j c_8 \le M_t^x \le 2^{j+1} c_8) \le 2^{-2j} c_8^{-2} \mathbb{E}(M_t^x)^2 \le 2^{-2j}.$$

Let j_1 be such that $\sum_{j \ge j_1} 2^{-j+1} c_8 \le c_7/2$. Then, by (3.12) and the last estimate,

$$\begin{aligned} c_7 &\le \mathbb{E} M_t^x \\ &\le (c_7/4)\mathbb{P}(0 < M_t^x \le c_7/4) + 2^{j_1+1} c_8 \mathbb{P}(c_7/4 \le M_t^x \le 2^{j_1+1} c_8) \\ &\quad + \sum_{j \ge j_1} 2^{j+1} c_8 \mathbb{P}(2^j c_8 \le M_t^x \le 2^{j+1} c_8) \\ &\le c_7/4 + 2^{j_1+1} c_8 \mathbb{P}(c_7/4 \le M_t^x \le 2^{j_1+1} c_8) + \sum_{j \ge j_1} 2^{j+1} c_8 2^{-2j} \\ &\le c_7/4 + 2^{j_1+1} c_8 \mathbb{P}(c_7/4 \le M_t^x \le 2^{j_1+1} c_8) + c_7/2, \end{aligned}$$

and, therefore, for $x \in V_1$,

$$\mathbb{P}(c_7/4 \le M_t^x \le 2^{j_1+1} c_8) \ge c_7 c_8^{-1} 2^{-j_1-3}.$$

Let $c_9 = c_7 c_8^{-1} 2^{-j_1-3}$. Assume that $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ are independent. Then, for $x \in V_1$,

$$\mathbb{P}(M_t^x \ge c_7/4, \widetilde{M}_t^x \ge c_7/4) \ge c_9^2. \tag{3.13}$$

Let K be the number of x such that $M_t^x \ge c_7/4$ and $\widetilde{M}_t^x \ge c_7/4$. Then

$$\sum_{x \in V} |M_t^x - \widetilde{M}_t^x| \le \sum_{x \in V} M_t^x + \sum_{x \in V} \widetilde{M}_t^x - K c_7/4 = 2n^d - K c_7/4.$$

Recall that $|V_1| \ge n^d/2$. By (3.13),

$$\mathbb{E}\left(\sum_{x\in V}|M_t^x - \widetilde{M}_t^x|\right) \le 2n^d - (n^d/2)c_9^2 c_7/4.$$

This completes the proof of (3.5).

Step 2. In this step, we will show that (3.5) holds (with a different constant) even if $\mathcal{M}_t$ and $\widetilde{\mathcal{M}}_t$ are not independent. More precisely, we will argue that there exist constants $c_1, c_2 \in (0,\infty)$ and $c_{10} < 2$, not depending on G, such that if $n \ge 1 \vee c_1\sqrt{d\log d}$ and $t \ge c_2 dn^2$, then for some coupling of $\mathcal{M}_t$ and $\widetilde{\mathcal{M}}_t$,

$$\mathbb{E}\left(\sum_{x\in V}|M_t^x - \widetilde{M}_t^x|\right) \le c_{10}|V|. \tag{3.14}$$

We will employ several families of WIMPs. Let $\{Z_t^j, t\ge 0\}_{j\ge1}$ and $\{\widetilde{Z}_t^j, t \ge 0\}_{j\ge1}$ be as in Definition 2.1. In particular, the jump times of all Z^j's and $\widetilde{Z}^j$'s are defined by the same family of Poisson processes $\{N^v\}_{v\in V}$. Let $\mathcal{M}_t$ and $\widetilde{\mathcal{M}}_t$ denote the mass processes corresponding to $\{N^v\}_{v\in V}$.

Let $\{X_t^j, t \ge 0\}_{j\ge1}$ be jointly distributed as $\{Z_t^j, t\ge 0\}_{j\ge1}$. Similarly, let $\{\widetilde{X}_t^j, t\ge 0\}_{j\ge1}$ be jointly distributed as $\{\widetilde{Z}_t^j, t\ge 0\}_{j\ge1}$. However, we make the family $\{X_t^j, t\ge 0\}_{j\ge1}$ independent of $\{\widetilde{X}_t^j, t\ge 0\}_{j\ge1}$. Let $\{\mathcal{R}_t, t\ge 0\}$ have the same distribution as $\{\mathcal{M}_t, t\ge 0\}$, and assume that $\mathcal{R}_t$ is driven by the same family of Poisson processes as $\{X_t^j, t\ge 0\}_{j\ge1}$. By analogy, let $\{\widetilde{\mathcal{R}}_t, t\ge 0\}$ have the same distribution as $\{\widetilde{\mathcal{M}}_t, t\ge 0\}$, and assume that $\widetilde{\mathcal{R}}_t$ is driven by the same family of Poisson processes as $\{\widetilde{X}_t^j, t\ge 0\}_{j\ge1}$. The processes $\mathcal{R}_t = \{R_t^x\}_{x\in V}$ and $\widetilde{\mathcal{R}}_t = \{\widetilde{R}_t^x\}_{x\in V}$ are independent.

Fix some $t > 0$ and integer $m > 0$. We find a maximal matching between (some) X^j's and (some) $\widetilde{X}^j$'s; that is, we find an asymmetric relation $\sim$ (a subset of $\{1,2,\dots,m\}^2$) such that $i\sim j$ only if $X_t^i = \widetilde{X}_t^j$. Moreover $i\sim j_1$ and $i\sim j_2$ implies $j_1 = j_2$ and, similarly, $i_1\sim j$ and $i_2 \sim j$ implies $i_1 = i_2$. Among all such relations $\sim$ we choose one of those that have the greatest number of matched pairs. Note that for every $x\in V$, either all i with $X_t^i = x$ are matched with some j, or all j with $\widetilde{X}_t^j = x$ are matched with some i (or both). Recall that $\sim$ depends on m and let r_m be the (random) number of matched pairs.

By the law of large numbers, a.s., for $x\in V$,

$$\lim_{m\to\infty}\frac{1}{m}\sum_{j=1}^m \mathbf{1}_{\{X_t^j = x\}} = R_t^x/|V|, \qquad \lim_{m\to\infty}\frac{1}{m}\sum_{j=1}^m \mathbf{1}_{\{\widetilde{X}_t^j = x\}} = \widetilde{R}_t^x/|V|.$$

This implies that

$$\lim_{m\to\infty}\left(\frac{1}{m}\sum_{j=1}^m \mathbf{1}_{\{X_t^j = x\}} - \frac{1}{m}\sum_{j=1}^m \mathbf{1}_{\{\widetilde{X}_t^j = x\}}\right) = \frac{1}{|V|}(R_t^x - \widetilde{R}_t^x),$$

and, a.s.,

$$\lim_{m\to\infty}\left(\frac{1}{m}(2m-r_m)\right)=\frac{1}{|V|}\sum_{x\in V}|R_t^x-\widetilde{R}_t^x|. \tag{3.15}$$

Hence, a.s.,

$$\frac{1}{|V|}\sum_{x\in V}|R_t^x-\widetilde{R}_t^x|=2-\lim_{m\to\infty}\frac{r_m}{m}. \tag{3.16}$$

Next we define a new relation $\approx$ (a subset of $\{1,2,\dots,m\}^2$). Recall that $t>0$ and $m>0$ are fixed. We will construct $\approx$ by adding pairs to this relation in a dynamic way. We start by letting $i\approx j$ if $i\sim j$ at time 0. Informally speaking, we match X_0^i and $\widetilde{X}_0^j$ if they are at the same vertex, and we try to match as many pairs as possible at the initial time. We wait until the first time $s_1>0$ when there exist i_1 and j_1 such that $i_1\not\approx j$ for all j, $i\not\approx j_1$ for all i, and $X_{s_1}^{i_1}=\widetilde{X}_{s_1}^{j_1}$. We add the pair (i_1,j_1) to the relation $\approx$. We proceed by induction. Given s_{k-1}, let $s_k>s_{k-1}$ be the first time when there exist i_k and j_k such that $i_k\not\approx j$ for all j, $i\not\approx j_k$ for all i (at times between s_{k-1} and s_k), and $X_{s_k}^{i_k}=\widetilde{X}_{s_k}^{j_k}$. We add the pair (i_k,j_k) to the relation $\approx$. We proceed in this way until time t. Let r_m^* be the number of matched pairs at time t.

We will find a lower bound for r_m^* in terms of r_m. Suppose that $i_1\sim j_1$. This implies that $X_t^{i_1}=\widetilde{X}_t^{j_1}$. Hence it is possible that $i_1\approx j_1$. In this case, a pair (i_1,j_1) that is in relation $\sim$ is also in relation $\approx$.

If $i_1\not\approx j_1$, then it must be the case that in the construction of the relation $\approx$, either X^{i_1} was matched with some $\widetilde{X}^{j_1^-}$ before time t, or $\widetilde{X}^{j_1}$ was matched with some $X^{i_1^+}$ before time t, or both. We will write $i\mathrel{\dot{\approx}}j$ if and only if $j\approx i$. Let

$$\begin{aligned} &i_{\min}^-\sim j_{\min}^-\cdots i_2^-\sim j_2^-\mathrel{\dot{\approx}}i_1^-\sim j_1^-\mathrel{\dot{\approx}}i_1\sim j_1\mathrel{\dot{\approx}}i_1^+\sim j_1^+\mathrel{\dot{\approx}}i_2^+\sim\\ &\quad\sim j_2^+\cdots i_{\max}^+\sim j_{\max}^+ \end{aligned} \tag{3.17}$$

be the maximal chain with the alternating structure that should be clear from the formula. The chain does not have to end with $j_{\max}^+$; it could end with $i_{\max}^+$. A similar remark applies to the left end of the chain. The minimal ratio of the number of pairs of integers in the chain which are in relation $\approx$ to the number of pairs of integers in the chain which are in relation $\sim$ is $1/2$.

Any two chains of the form given in (3.17) are either identical or disjoint. Recall that if $i_1\sim j_1$, then either $i_1\approx j_1$ or the pair (i_1,j_1) is an element of a chain as in (3.17). It follows that

$$r_m^*\ge r_m/2. \tag{3.18}$$

Recall that $t>0$ is fixed. If $i\approx j$, then let $\sigma^X(i,j)=\inf\{t\ge 0\colon X_t^i=\widetilde{X}_t^j\}$. Otherwise, let $\sigma^X(i,j)=t$.

Recall WIMPs $\{Z_t^j, t \geq 0\}_{j\geq 1}$ and $\{\widetilde{Z}_t^j, t \geq 0\}_{j\geq 1}$. Let a relation $\triangleq$ be defined relative to these WIMPs in exactly the same manner as $\approx$ was defined for $\{X_t^j, t \geq 0\}_{j\geq 1}$ and $\{\widetilde{X}_t^j, t \geq 0\}_{j\geq 1}$. In other words, $\triangleq$ matches colliding particles of type Z^i with $\widetilde{Z}^j$ as soon as the collisions occur, with the restriction that each particle is matched with at most one other particle. If $i \triangleq j$, then let $\sigma^Z(i, j) = \inf\{t \geq 0 : Z_t^i = \widetilde{Z}_t^j\}$. Otherwise, let $\sigma^Z(i, j) = t$.

Recall that $(i, j) \in \approx$ is equivalent to $i \approx j$. It is easy to see that the distribution of the family

$$\big(\{(X_s^i, \widetilde{X}_s^j), 0 \leq s \leq \sigma^X(i, j)\}_{(i,j)\in\approx}, \{(X_s^i, \widetilde{X}_s^j), 0 \leq s \leq t\}_{(i,j)\notin\approx}\big)$$

is the same as that of

$$\big(\{(Z_s^i, \widetilde{Z}_s^j), 0 \leq s \leq \sigma^Z(i, j)\}_{(i,j)\in\triangleq}, \{(Z_s^i, \widetilde{Z}_s^j), 0 \leq s \leq t\}_{(i,j)\notin\triangleq}\big)$$

because the jump times of the processes in each family are determined by independent Poisson processes at vertices of the graph. Let r_m^Z be the number of pairs in the relation $\triangleq$. We see that the distributions of r_m^Z and r_m^* are identical.

If $\sigma^Z(i, j) < t$, then we let $\widehat{Z}_s = \widetilde{Z}_s$ for $s \in [0, \sigma^Z(i, j))$ and $\widehat{Z}_s = Z_s$ for $s \geq \sigma^Z(i, j)$. Note that the distribution of the family $\{\widehat{Z}_t^j, t \geq 0\}_{j\geq 1}$ is the same as that of the family $\{\widetilde{Z}_t^j, t \geq 0\}_{j\geq 1}$. If $i \triangleq j$, then $Z_t^i = \widehat{Z}_t^j$. We have

$$\lim_{m\to\infty}\left(\frac{1}{m}(2m - r_m^Z)\right) = \frac{1}{|V|}\sum_{x\in V}|M_t^x - \widetilde{M}_t^x|,$$

for the same reason that (3.15) holds. Therefore, using (3.5), (3.16), (3.18) and the equality of the distributions of r_m^Z and r_m^*, we obtain for $n \geq 1 \vee c_1\sqrt{d\log d}$ and $t \geq c_2 dn^2$,

$$\mathbb{E}\left(\frac{1}{|V|}\sum_{x\in V}|M_t^x - \widetilde{M}_t^x|\right) \tag{3.19}$$

$$\begin{aligned}
&= 2 - \mathbb{E}\left(\lim_{m\to\infty}\frac{r_m^Z}{m}\right) &(3.20)\\
&\leq 2 - \mathbb{E}\left(\lim_{m\to\infty}\frac{r_m}{2m}\right)\\
&= 2 - \mathbb{E}\left(\lim_{m\to\infty}\frac{r_m}{m}\right) + \frac{1}{2}\mathbb{E}\left(\lim_{m\to\infty}\frac{r_m}{m}\right)\\
&= \mathbb{E}\left(\frac{1}{|V|}\sum_{x\in V}|R_t^x - \widetilde{R}_t^x|\right) + \frac{1}{2}\left(2 - \mathbb{E}\left(\frac{1}{|V|}\sum_{x\in V}|R_t^x - \widetilde{R}_t^x|\right)\right)\\
&= 1 + \frac{1}{2}\mathbb{E}\left(\frac{1}{|V|}\sum_{x\in V}|R_t^x - \widetilde{R}_t^x|\right) \leq 1 + c_4/2.
\end{aligned}$$

This proves (3.14).

Step 3. The process $\mathcal{M}_t$ is "additive" in the following sense. Suppose that $\mathcal{M}_t$ and $\widetilde{\mathcal{M}}_t$ are driven by the same family of Poisson processes N^v. Let $\widehat{\mathcal{M}}_0 = \mathcal{M}_0 + \widetilde{\mathcal{M}}_0$, and suppose that $\widehat{\mathcal{M}}_t$ is also driven by the same family of Poisson processes N^v. Then $\widehat{\mathcal{M}}_t = \mathcal{M}_t + \widetilde{\mathcal{M}}_t$ for all t, a.s.

Fix $t = c_2 dn^2$ and suppose that $\mathcal{M}_t$ and $\widetilde{\mathcal{M}}_t$ are driven by the same family of Poisson processes N^v. Let

$$\begin{aligned}
\mathcal{M}_t^+ &= (\mathcal{M}_t - \widetilde{\mathcal{M}}_t) \vee 0, \\
\widetilde{\mathcal{M}}_t^+ &= (\widetilde{\mathcal{M}}_t - \mathcal{M}_t) \vee 0, \\
\mathcal{M}_t^c &= (\mathcal{M}_t - \mathcal{M}_t^+) = (\widetilde{\mathcal{M}}_t - \widetilde{\mathcal{M}}_t^+).
\end{aligned}$$

The process $\mathcal{M}_t^c$ represents the maximum matching mass at every site, and processes $\mathcal{M}_t^+$ and $\widetilde{\mathcal{M}}_t^+$ represent the excesses of $\mathcal{M}_t$ and $\widetilde{\mathcal{M}}_t$ (if any) above the common mass. Suppose that all these processes are driven by the same family of Poisson processes N^v after time t. Then for every $s \geq t$,

$$\mathcal{M}_s = \mathcal{M}_s^c + \mathcal{M}_s^+, \qquad \widetilde{\mathcal{M}}_s = \mathcal{M}_s^c + \widetilde{\mathcal{M}}_s^+.$$

By the Markov property applied at time $t = c_2 dn^2$ and (3.14) applied to $\mathcal{M}_s^+$ and $\widetilde{\mathcal{M}}_s^+$, we obtain for $s \geq 2c_2 dn^2$,

$$\mathbb{E}\Bigl(\sum_{x \in V} \bigl|(M^+)_s^x - (\widetilde{M}^+)_s^x\bigr| | \mathcal{F}_t\Bigr) \leq c_{10} \frac{1}{2} \sum_{x \in V} \bigl((M^+)_t^x + (\widetilde{M}^+)_t^x\bigr).$$

Hence

$$\begin{aligned}
\mathbb{E}\Bigl(\sum_{x \in V} \bigl|(M^+)_s^x - (\widetilde{M}^+)_s^x\bigr|\Bigr) &\leq (c_{10}/2)\mathbb{E}\Bigl(\sum_{x \in V} \bigl((M^+)_t^x + (\widetilde{M}^+)_t^x\bigr)\Bigr) \\
&= (c_{10}/2)\mathbb{E}\Bigl(\sum_{x \in V} |M_t^x - \widetilde{M}_t^x|\Bigr) \\
&\leq (c_{10}/2) c_{10} |V|.
\end{aligned}$$

An inductive argument applied at times t of the form $t = jc_2 dn^2$, $j \geq 2$, yields for $s \geq (j+1)c_2 dn^2$,

$$\mathbb{E}\Bigl(\sum_{x \in V} \bigl|(M^+)_s^x - (\widetilde{M}^+)_s^x\bigr|\Bigr) \leq (c_{10}/2)^j c_{10} |V|.$$

This implies (3.4) and completes the proof. $\square$

REMARK 3.7. (i) Let $\|\cdot\|_{\mathrm{TV}}$ denote the total variation distance, and let the mixing time for the random walk on G be defined by

$$\mathcal{T} = \inf\Bigl\{t \geq 0 : \sup_\mu \|\mu P_t - \pi\|_{\mathrm{TV}} \leq 1/4\Bigr\},$$

where π stands for the stationary distribution, and P_t denotes the transition kernel. See [28], Chapter 4, for these definitions and various results on mixing times.

Consider the graph C_n^d for some $n, d \geq 3$. For this graph, $\alpha(t)$ defined in (3.1) is equal to $1/2$ for t of order n^d. Theorem 3.6 shows that the left-hand side of (3.4) is bounded by $n^d/2$ for t of order n^2, thus greatly improving (3.2) in the case $G = C_n^d$. Since the mixing time for random walk on C_n^d is of the order n^2, the bound in Theorem 3.6 cannot be improved in a substantial way. Recall $\mathbf{P}$ defined before Theorem 3.4.

CONJECTURE. *The mixing time for the random walk corresponding to* $\mathbf{P}$ *should give the optimal bound in* (3.2).

A support to our conjecture is lent by the recent proof (see [8]) of the "Aldous spectral gap conjecture," saying that the "interchange process" and the corresponding random walk have the same spectral gap.

(ii) The proof of Theorem 3.6 depends on the assumption that $G = C_n^d$ only at one point, namely, the estimate

$$\begin{aligned} &\mathbb{P}(Z_t = \widetilde{Z}_t = x \mid Z_t = y, \widetilde{Z}_0 = z) \\ &\quad \leq c_0 \mathbb{P}(Z_t = x \mid Z_t = y) \mathbb{P}(\widetilde{Z}_t = x \mid \widetilde{Z}_0 = z) \end{aligned} \tag{3.21}$$

is derived using properties of C_n^d in an essential way. In other words, if a similar estimate can be obtained for some other family of graphs, the proof of the theorem would apply in that case. It is not hard to construct examples showing that there is no universal constant c_0 such that (3.21) holds for all finite graphs G, all $x, y, z \in V$ and all $t > 0$. Hence, any generalization of Theorem 3.6 has to be limited to a subfamily of finite graphs or come with a different proof.

We now present very elementary properties of the meteor process.

PROPOSITION 3.8. *Let T_t^v denote the time of the last jump of N^v on the interval* $[0, t]$, *with the convention that $T_t^v = -1$ if there were no jumps on this interval. Let $U(v) = \{v\} \cup \{x \in V : x \leftrightarrow v\}$.*

(i) *Assume that $M_0^v + M_0^x > 0$ for a pair of adjacent vertices v and x. Then, almost surely, for all $t \geq 0$, $M_t^v + M_t^x > 0$.*

(ii) *Let R_t be the number of pairs (x, v) such that $x \leftrightarrow v$ and $M_t^v + M_t^x = 0$. The process R_t is nonincreasing, a.s.*

(iii) *Assume that $M_0^x > 0$ for $x \in U(v) \setminus \{v\}$. Then, a.s., $M_t^v = 0$ if and only if one of the following conditions holds*: (a) $T_t^v = \max\{T_t^x : x \in U(v)\} > -1$ *or* (b) $M_0^v = 0$ *and* $\max\{T_t^x : x \in U(v) \setminus \{v\}\} = -1$.

(iv) *Suppose that the process $\{\mathcal{M}_t, t \geq 0\}$ is in the stationary regime, that is, its distribution at time* 0 *is the stationary distribution Q. Then $M_t^v + M_t^x > 0$ for all $t \geq 0$ and all pairs of adjacent vertices v and x, a.s.*

(v) *Recall from Remark 3.3 the stationary mass process* $\{\mathcal{M}_t, t \in \mathbb{R}\}$ *and the corresponding Poisson processes* $\{N_t^v, t \in \mathbb{R}\}$, $v \in V$. *Let* T^v *denote the time of the last jump of* N^v *on the interval* $(-\infty, 0]$, *and note that* T^v *is well defined for every* v *because such a jump exists, a.s. Then, a.s.,* $M_0^v = 0$ *if an only if* $T^v = \max\{T^x : x \in U(v)\}$.

PROOF. (i) Suppose to the contrary that $M_t^v + M_t^x = 0$ for some $x \leftrightarrow v$ and $t > 0$. The value of $M_t^v + M_t^x$ can change only when one of the processes N^y, $y \in U_1 := U(x) \cup U(v)$, has a jump. Note that U_1 is a finite set. It follows that the union of jump times of all processes N^y, $y \in U_1$, does not have accumulation points. Moreover, jumps of different processes N^y in this family never occur at the same time, a.s. Let T be the infimum of times such that $M_t^v + M_t^x = 0$. Then $M_s^v + M_s^x > 0$ for all $s < T$ and $M_{T-}^v + M_{T-}^x > 0$. Suppose without loss of generality that $M_{T-}^v > 0$. If N^v has a jump at T, then $M_T^x > 0$, a contradiction. If N^y has a jump at T for some $y \in U_1$, then N^v does not have a jump at T and, therefore, $M_T^v > 0$, also a contradiction. We conclude that the assumption that $M_t^v + M_t^x = 0$ for some $x \leftrightarrow v$ and $t > 0$ is false.

(ii) For $x \leftrightarrow v$, let $R_t^{x,v}$ be 1 if $M_t^v + M_t^x = 0$ and 0 otherwise. This process is nonincreasing, by part (i). Since $R_t = \sum_{x,v \in V, x \leftrightarrow v} R_t^{x,v}$, it follows that R_t is nonincreasing.

(iii) If $M_0^v = 0$ and $\max\{T_t^x : x \in U(v) \setminus \{v\}\} = -1$, then processes N^x, $x \in U(v) \setminus \{v\}$, do not jump on the interval $[0, t]$. Hence, $M_s^v = 0$ for all $s \in [0, t]$. In particular, $M_t^v = 0$. We will assume that $\max\{T_t^x : x \in U(v) \setminus \{v\}\} > -1$ in the rest of the proof.

Suppose that $T_t^v = \max\{T_t^x : x \in U(v)\} > -1$. Then $M_{T_t^v}^v = 0$. Since processes N^x, $x \leftrightarrow v$, do not have jumps on the interval $[T_t^v, t]$, we must have $M_s^v = 0$ for all $s \in [T_t^v, t]$. Hence, $M_t^v = 0$.

Suppose that $T_t^v < \max\{T_t^x : x \in U(v)\}$ and let y be such that $T_t^y = \max\{T_t^x : x \in U(v)\} > -1$ and $y \leftrightarrow v$. By part (i), either $M_{T_t^y-}^v > 0$ or $M_{T_t^y-}^y > 0$ (or both). If $M_{T_t^y-}^v > 0$, then $M_s^v > 0$ for all $s \in [T_t^y, t]$ because N^v does not jump on this interval. If $M_{T_t^y-}^y > 0$, then $M_{T_t^y}^v > 0$ and, therefore, $M_s^v > 0$ for all $s \in [T_t^y, t]$ because N^v does not jump on this interval. We see that in either case, $M_t^v > 0$.

(iv) Since V is finite, there exists a sequence $(x_1, x_2, \dots, x_n)$ of vertices of G such that $x_j \leftrightarrow x_{j+1}$ for $1 \le j \le n-1$, $x_n \leftrightarrow x_1$, and the sequence contains all vertices in V. The vertices x_j's are not necessarily distinct.

Let A_i be the event that processes N^v, $v \in V$, have $2n$ jumps in the time interval $[i, i+1)$, and the jumps occur at the following vertices in the following order: $x_1, x_2, \dots, x_n, x_1, x_2, \dots, x_n$. It is easy to see that if A_i occurs, then there is only one vertex v with $M_{i+1}^v = 0$; specifically, $v = x_n$. Hence if A_i occurs, then $R_{i+1} = 0$ and, by part (ii), $R_t = 0$ for $t \ge i+1$. Events A_i are independent and have positive probability so the probability of $A_1 \cup A_2 \cup \cdots \cup A_m$ is bounded below by $1 - p^m$ for some $p < 1$. It follows that $\mathbb{P}(R_{m+1} = 0) \geq 1 - p^m$ for $m \geq 1$. This and stationarity imply that $\mathbb{P}(R_0 = 0) = \mathbb{P}(R_{m+1} = 0) = 1$ for $m \geq 1$.

(v) It follows from part (iv) that $M_0^v + M_0^x > 0$ for all pairs of adjacent vertices v and x, a.s. Hence $M_k^v + M_k^x > 0$ for all $k \in \mathbb{Z}$ and, therefore, $M_t^v + M_t^x > 0$ for all pairs of adjacent vertices v and x and all $t \in \mathbb{R}$, a.s. Now we can apply the same reasoning as in the proof of case (a) in part (iii). $\square$

4. Meteor craters in circular graphs. This section is devoted to meteor processes on circular graphs. Recall that C_k denotes the circular graph with k vertices, $k \geq 2$.

We will say that there is a *crater* at the site j at time t if $M_t^j = 0$. Craters are special features of the meteor process for a number of reasons. First, the mass at a crater has the minimum possible value. Second, we expect that the distribution of mass M_t^j is a mixture of an atom at 0 and a distribution with a continuous density. Third, given the distribution of mass $\mathcal{M}_s$ at all sites at time s and positions of all craters at times $t \in [s, u]$, we can determine the mass process $\{\mathcal{M}_t, t \in [s, u]\}$. For these reasons, we find it interesting to study the distribution of craters. An easy argument (see the proof of Theorem 4.1) shows that the concept of a crater is essentially equivalent to a *peak* in a random (uniform) permutation.

The research on peaks and other related permutation statistics, such as valleys, descents and runs has a very long history. For a review of some related literature, see the introduction to [12]; the authors trace the beginning of this line of research to the nineteenth century. However, the research in this area seems to have a number of separate lines, because the authors of [12] do not cite [10] or [26, 27]. In view of this disconnected nature of the literature we are not sure whether we were able to trace all the existing results in the area that are relevant to our paper.

There are (at least) three natural probabilistic questions that have to do with craters. The first one is concerned with the probability of a given pattern of craters. This is equivalent, more or less, to the question about the asymptotic frequency of a given pattern of craters in a very large cyclic graph C_k. We will provide formulas for two specific crater "patterns" in Theorems 4.1 and 4.2. It is possible that both results could be derived from [26, 27], but the style of those old papers may be hard to follow for the modern reader. We will base our proofs on the combinatorial results in [3]. The results in [3] could be used to derive more advanced theorems on craters that go beyond the scope of this paper.

The second question is about fluctuations of the number of copies of a pattern. There are a number of combinatorial versions of the central limit theorem for permutation statistics; see, for example, [10, 12] and references therein. We will state a theorem that appeared in [5], and we will provide a new short proof based on classical probabilistic tools and our meteor process.

Finally, there is a question of large deviations for the crater process. We will not provide a standard large deviations result, but we will prove a theorem on the most likely configuration of craters assuming that there are very few of them.

Consider the meteor process on C_k, and assume that $\sum_{j=1}^{k} M_0^j = k$. For $n \geq 1$ and $k \geq n+4$, let F_n^k be the event that $M_0^i > 0$ for $i = 3, 4, \ldots, n+2$ and $M_0^2 = M_0^{n+3} = 0$. In other words, F_n^k is the event that 3 is the starting point of a maximal sequence of vertices which are not craters at time 0 and that sequence has length n. We let F_0^k be the event that $M_0^2 = 0$.

For $n \geq 1$ and $k \geq n+3$, let $\widehat{F}_n^k$ be the event that $M_0^i > 0$ for $i = 3, \ldots, n+2$. In other words, $\widehat{F}_n^k$ is the event that sites $3, \ldots, n+2$ are not craters at time 0, but this sequence does not have to be maximal.

THEOREM 4.1. *Consider the meteor process on C_k in the stationary regime; that is, assume that the distribution of $\mathcal{M}_0$ is the stationary measure Q. We have*

$$p_0 := \mathbb{P}(F_0^k) = 1/3, \qquad k \geq 3, \tag{4.1}$$

$$p_n := \mathbb{P}(F_n^k) = \frac{n(n+3)2^{n+1}}{(n+4)!}, \qquad n \geq 1, k \geq n+4, \tag{4.2}$$

$$\widehat{p}_n := \mathbb{P}(\widehat{F}_n^k) = \frac{2^{n+1}}{(n+2)!}, \qquad n \geq 1, k \geq n+3. \tag{4.3}$$

PROOF. Recall from Remark 3.3 the stationary mass process $\{\mathcal{M}_t, t \in \mathbb{R}\}$ and the corresponding Poisson processes $\{N_t^j, t \in \mathbb{R}\}$, $j = 1, \ldots, k$, defined on the whole real time-line. As in Proposition 3.8(v), we let T^j denote the time of the last jump of N^j on the interval $(-\infty, 0]$. According to Proposition 3.8(v), $M_0^j = 0$ if and only if $T^{j-1} < T^j > T^{j+1}$.

Note that $T^m \neq T^j$ if $m \neq j$, a.s. Let $a_1 \cdots a_k$ be the random permutation of $\{1, 2, \ldots, k\}$ defined by the condition $a_j < a_m$ if and only if $T^j < T^m$, for all j and m. It is clear that $a_1 \cdots a_k$ is the uniform random permutation of $\{1, 2, \ldots, k\}$.

We say that j is a *peak* (of the permutation $a_1 \cdots a_k$) if $a_{j-1} < a_j > a_{j+1}$. Hence $M_0^j = 0$ if and only if a_j is a peak.

By symmetry, any of the random numbers a_{j-1}, a_j and a_{j+1} is the largest of the three with the same probability. Hence the probability that $a_{j-1} < a_j > a_{j+1}$ is $1/3$. This proves (4.1).

The event F_n^k holds if and only if in the initial part $a_1 \cdots a_{n+4}$ of the permutation $a_1 \cdots a_k$, there are exactly two peaks at 2 and $n+3$. It is clear that the probability of this event is the same if $a_1 \cdots a_{n+4}$ is a random uniform permutation of $\{1, \ldots, n+4\}$ with the same peak set. Recall that the number of permutations of $\{1, \ldots, n+4\}$ is $(n+4)!$. We now see that (4.2) follows from Theorems 1 and 10 in [3]. Note that we are concerned with permutations of size $n+4$ while the two cited theorems in [3] count permutations of size n. This explains the shift of size 4 in the corresponding formulas in our paper and [3].

Finally, we will prove (4.3). The event $\widehat{F}_n^k$ holds if an only if there are no peaks in the part $a_2 \cdots a_{n+3}$ of the permutation $a_1 \cdots a_k$. The probability of this event is the same if $a_2 \cdots a_{n+3}$ is a random uniform permutation of $\{1, \dots, n+2\}$ with no peaks. Formula (4.3) follows from Proposition 2 in [3], with a shift of size 2 between the corresponding formulas in our paper and [3]. $\square$

The results in [3] provide an effective tool for calculating various distributions related to crater positions. We ask the interested reader to consult that paper for the general theory. We will provide here another explicit probabilistic formula based on combinatorial results from [3].

THEOREM 4.2. *Consider the meteor process on C_k in the stationary regime; that is, assume that the distribution of $\mathcal{M}_0$ is the stationary measure Q. For $i, j \geq 1$ and $k \geq i + j + 5$, let $A_{i,j}^k$ be the event that $M_0^2 = M_0^{i+3} = M_0^{i+j+4} = 0$ and $M_0^n > 0$ for $n = 3, 4, \dots, i+2, i+4, \dots, i+j+3$. In other words, $A_{i,j}^k$ is the event that 2 is a crater and the gaps between this crater and the next two craters have sizes i and j. We have*

$$\begin{aligned}\mathbb{P}(A_{i,j}^k) = \frac{2^{i+j}}{(i+j+5)!}\biggl[(i+j+4)\biggl(j\binom{i+j+1}{i-1} + (j+1)\binom{i+j+1}{i} \\ + (i+1)\binom{i+j+1}{i+1} + i\binom{i+j+1}{i+2} \\ - 2(i+j+1)\biggr) \\ + ij\binom{i+j+4}{i+2}\biggr].\end{aligned} \tag{4.4}$$

PROOF. The theorem follows from Theorems 9 and 12 of [3]. The argument is totally analogous to that in the proof of Theorem 4.1 so we leave the details to the reader. We just note that one should take $m = i + 3$ and $n = i + j + 5$ in Theorem 12 of [3]. $\square$

REMARK 4.3. (i) If craters occurred in the i.i.d. manner, then the distribution of the distance between consecutive craters would have been geometric, with the tail decaying exponentially. This is not the case. By the Stirling approximation,

$$p_n = \frac{n(n+3)2^{n+1}}{(n+4)!} \sim \frac{n(n+3)2^{n+1}e^{n+4}}{(n+4)^{n+4}\sqrt{2\pi(n+4)}}.$$

Hence, p_n converges to 0 at a rate faster than exponential; specifically, $\log p_n \approx -n \log n$.

(ii) Despite remark (i), the crater process is "partly" memoryless. Consider the crater distribution at time 0 assuming that the mass process $\{\mathcal{M}_t, t \in \mathbb{R}\}$ is in the stationary regime. The event that there is a crater at site j depends only on the Poisson processes N^n for $n = j-1, j, j+1$, by Proposition 3.8(v). Hence, the events $\{M_0^j = 0\}$ for $j = 1 + 3m$, $m \in \mathbb{Z}$, $1 \le j \le k-2$, form a sequence of Bernoulli trials (are i.i.d.). It follows that the gap between the first and second craters in this sequence has an approximately geometric tail, for large k. The same observation holds for two similar sequences of sites, namely for those indexed by $j = 2 + 3m$, $m \in \mathbb{Z}$, $1 \le j \le k-2$, and those indexed by $j = 3m$, $m \in \mathbb{Z}$, $1 \le j \le k-2$. However, the three sequences of Bernoulli trials are highly dependent.

(iii) It is natural to ask for the distribution of the number of consecutive sites with nonzero mass following a crater. This somewhat informal statement can be translated into a rigorous question about the conditional probability of F_n^k given that $M_0^2 = 0$. The answer is $p_n/p_0 = 3n(n+3)2^{n+1}/(n+4)!$. In other words, a crater is followed by exactly n consecutive sites with nonzero mass with probability $3n(n+3)2^{n+1}/(n+4)!$.

(iv) Remarks (i) and (ii) make it clear that the we should not expect independence between the lengths of consecutive stretches of sites with nonzero mass. More precisely, one can easily check that, in general, for large k,

$$\frac{1}{p_0}\mathbb{P}(A_{i,j}^k) \ne \frac{1}{p_0}\mathbb{P}(F_i^k)\frac{1}{p_0}\mathbb{P}(F_j^k).$$

Curiously, for $j \ge 1$ and $k \ge j+7$,

$$\frac{1}{p_0}\mathbb{P}(A_{2,j}^k) = \frac{1}{p_0}\mathbb{P}(F_2^k)\frac{1}{p_0}\mathbb{P}(F_j^k). \tag{4.5}$$

Hence, if there are exactly two noncraters between two consecutive craters, then this event gives no information about the length of the next stretch of sites with nonzero masses. Formula (4.5) follows from (4.2) and (4.4) by direct calculation. Formula (4.5) does not seem to hold if 2 is replaced by any other integer $i \ge 1$, $i \ne 2$. We offer an informal explanation of (4.5). Suppose that there is a crater at site 5. Then there is no crater at site 4. The distribution of craters at sites $5, 6, \dots$ is determined by Poisson processes at sites $4, 5, \dots$. If we have extra information that there is a crater at site 2, then this tells us only that the latest meteor hit among the sites 1, 2 and 3 occurred at site 2. Since the Poisson processes at sites 1, 2 and 3 are independent of those at sites $4, 5, \dots$, the information that 2 is a crater has no predictive value for craters to the right of 5.

When translated into the language of permutation peaks, the condition discussed in the last paragraph becomes that there are exactly two nonpeaks between any two consecutive peaks. Interestingly, exactly the same condition came up as part of a conjecture in [3] about the equidistribution of peaks in permutations. This part of the conjecture was recently proved by Kasraoui in [25]. Is there some deeper connection between this result and equation (4.5)?

(v) Formula (4.3) is extremely easy to prove; see the counting argument in the proof of Proposition 2 in [3]. We will derive the harder formula (4.2) from the easier formula (4.3) in an informal way. It has been shown in [6], a follow-up paper, that stationary distributions on C_k converge to a stationary distribution for the meteor process on $\mathbb{Z}$, in an appropriate sense. It is easy to see that for the meteor process on $\mathbb{Z}$,

$$\widehat{p}_n = p_n + 2p_{n+1} + 3p_{n+2} + \cdots.$$

We take the inverse of this linear transformation to see that

$$p_n = \widehat{p}_n - 2\widehat{p}_{n+1} + \widehat{p}_{n+2}.$$

This and (4.3) imply that

$$p_n = \frac{2^{n+1}}{(n+2)!} - 2\frac{2^{n+2}}{(n+3)!} + \frac{2^{n+3}}{(n+4)!} = \frac{n(n+3)2^{n+1}}{(n+4)!}.$$

THEOREM 4.4. *We have*

$$\sum_{n=0}^{\infty} p_n = 2/3, \tag{4.6}$$

$$\sum_{n=0}^{\infty} np_n = 2/3. \tag{4.7}$$

PROOF. Our argument is based on power series expansions derived by Mathematica [29]. The following power series converges for all real x,

$$\sum_{n=1}^{\infty} \frac{2n(n+3)}{(n+4)!} x^{n+4} = \frac{2x^3}{3} + 2x^2 + 2e^x(x-2)^2 - 8.$$

From this, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} p_n &= 1/3 + \sum_{n=1}^{\infty} p_n = 1/3 + \sum_{n=1}^{\infty} \frac{n(n+3)}{(n+4)!} 2^{n+1} \\ &= 1/3 + 2^{-4} \sum_{n=1}^{\infty} \frac{2n(n+3)}{(n+4)!} 2^{n+4} \\ &= 1/3 + 2^{-4}\left(\frac{2 \cdot 2^3}{3} + 2 \cdot 2^2 + 2e^2(2-2)^2 - 8\right) = \frac{2}{3}.\end{aligned}$$

A similar calculation yields

$$\sum_{n=1}^{\infty} \frac{2n^2(n+3)}{(n+4)!} x^{n+4} = 2e^x(x^3 - 6x^2 + 16x - 16) - \frac{2}{3}(x^3 + 6x^2 - 48)$$

and

$$\sum_{n=0}^{\infty} np_n = \sum_{n=1}^{\infty} \frac{n^2(n+3)}{(n+4)!} 2^{n+1} = 2^{-4} \sum_{n=1}^{\infty} \frac{2n^2(n+3)}{(n+4)!} 2^{n+4}$$
$$= 2^{-4}\left(2e^2(2^3 - 6 \cdot 2^2 + 16 \cdot 2 - 16) - \frac{2}{3}(2^3 + 6 \cdot 2^2 - 48)\right) = \frac{2}{3}.$$

This completes the proof. □

REMARK 4.5. (i) The reader may be puzzled by (4.6) since the probabilities do not add up to 1. This sequence of probabilities does not represent all events in a partition of a probability space. For the meteor process on $\mathbb{Z}$ constructed in [6], the probabilities p_n represent only the events that a given vertex has no mass or it is the starting point of a sequence of consecutive vertices, all with positive masses. It is also possible for a vertex to be an interior point of a sequence of consecutive vertices with positive masses. It follows from (4.6) that the last event has probability $1/3$.

(ii) We will present a simple heuristic proof of (4.6) and (4.7) based on the meteor process on $\mathbb{Z}$ constructed in [6]. Recall from (4.1) that $p_0 = 1/3$. The number of starting points of sequences of consecutive vertices with positive masses must be the same as the number of craters, since such vertices are never adjacent, by Proposition 3.8(iv). Hence, $\sum_{n=1}^{\infty} p_n = 1/3$, implying (4.6). The sum $\sum_{n=1}^{\infty} np_n$ represents the proportion of noncraters so it must be equal to $2/3$ because the proportion of craters is $p_0 = 1/3$.

One can ask not only how often a given configuration of craters occurs in a very large circular graph C_k but also what the random fluctuations are. We will prove a central limit theorem to shed some light on this problem. To match well the existing literature, our formulation will be more general than necessary for the purpose of describing the configuration of craters.

Consider the meteor process on C_k in the stationary regime; here and later in this section this means that the distribution of $\mathcal{M}_0$ is the stationary measure Q. Recall from Remark 3.3 that the stationary mass process $\{\mathcal{M}_t, t \in \mathbb{R}\}$ and the corresponding Poisson processes $\{N_t^m, t \in \mathbb{R}\}$, $m = 1, \dots, k$, are defined on the whole real time-line. As in Proposition 3.8(v), we let T^m denote the time of the last jump of N^m on the interval $(-\infty, 0]$. A permutation $\mathbf{a} = a_1 a_2 \cdots a_n$ of $\{1, \dots, n\}$ will be called a *pattern*. We will denote finite families of patterns by $\mathcal{A} = \{\mathbf{a}^1, \dots, \mathbf{a}^m\}$. We will not assume that all patterns in $\mathcal{A}$ have the same length. We will say that $\mathcal{A}$ occurs at j if for some $\mathbf{a}^r = a_1^r \cdots a_{n_r}^r \in \mathcal{A}$, we have $T^{j+i-1} < T^{j+m-1}$ if and only if $a_i^r < a_m^r$ for all $1 \le i, m \le n_r$.

According to Proposition 3.8(v), $M_0^m = 0$ if an only if $T^{m-1} < T^m > T^{m+1}$. Hence, any finite configuration of craters can be represented as a family of patterns.

THEOREM 4.6 ([5, 11]). *Consider the meteor process on C_k in the stationary regime. Fix a family of patterns $\mathcal{A}$, and let N be the number of sites in C_k where $\mathcal{A}$ occurs. Then there exist $\mu, \sigma > 0$ such that $(N - k\mu)/\sigma\sqrt{k}$ converges in distribution to the standard normal random variable as $k \to \infty$.*

PROOF. We will supply a proof that is shorter than that in [5] or [11], Example 6.2, and illustrates well the power of the meteor representation of craters and other patterns.

Let $\{U^j, j \in \mathbb{Z}\}$ be i.i.d. exponential random variables with mean 1. Note that for any fixed k, the distribution of $\{U^j, 1 \leq j \leq k\}$ is the same as that of $\{-T^j, 1 \leq j \leq k\}$, where T^j's are defined relative to C_k. Let ξ_m be the indicator random variable of the occurrence of $\mathcal{A}$ at the mth site in $\{U^j, j \in \mathbb{Z}\}$. In other words, $\xi_j = 1$ if and only if for some $\mathbf{a}^r = a_1^r \cdots a_{n_r}^r \in \mathcal{A}$, we have $U^{j+i-1} < U^{j+m-1}$ if and only if $a_i^r < a_m^r$ for all $1 \leq i, m \leq n_r$. Otherwise, $\xi_j = 0$.

It is clear that the process $\{\xi_j, j \in \mathbb{Z}\}$ is stationary.

Let b be the length of the longest pattern in $\mathcal{A}$. If $|j - m| > b$, then the occurrence of $\mathcal{A}$ at site j is independent of the occurrence of $\mathcal{A}$ at site m, since U^n's are independent. In other words, if $|j - m| > b$, then ξ_j and ξ_m are independent. This implies that the process $\{\xi_j, j \in \mathbb{Z}\}$ is φ-mixing in the sense of [4], Section 20. The central limit theorem holds for $\sum_{j=1}^{k-b} \xi_j$, according to [4], Theorem 20.1. Let N' be the number of sites $1 \leq j \leq k - b$ in C_k where $\mathcal{A}$ occurs, and note that N' has the same distribution as $\sum_{j=1}^{k-b} \xi_j$. Hence, the central limit theorem holds for N'. Since N and N' differ by at most b, the theorem follows. $\square$

REMARK 4.7. Theorem 20.1 of [4] not only yields the central limit theorem for N in Theorem 4.6 but also provides an effective algorithm for calculating μ and σ. To compute the values of these parameters, one has to find $\mathbb{E}\xi_1$ and $\mathbb{E}(\xi_1\xi_m)$ for all m. This is equivalent to counting the corresponding permutations of length at most $2b$ [because we have $\mathbb{E}(\xi_1\xi_m) = \mathbb{E}\xi_1\mathbb{E}\xi_m = (\mathbb{E}\xi_1)^2$ for $|1 - m| > b$]. For very small b, the counting can be done directly. For moderate b, formulas such as those in [3] may be helpful, depending on the family of patterns $\mathcal{A}$.

Craters represent sites that were hit by a meteor more recently than their nearest neighbors. We will now state a result about the locations of the sites such that both of its neighbors were hit by meteors more recently than the given site. Our result is partly motivated by a technical application later in this section.

Recall that, according to Proposition 3.8(v), m is a crater if an only if $T^{m-1} < T^m > T^{m+1}$. We will say that m is a *mound* if an only if $T^{m-1} > T^m < T^{m+1}$. Note that as we move along the graph C_k, we will encounter an alternating sequence of craters and mounds, separated by stretches of sites that are neither. The craters and mounds correspond to the local maxima and minima of the function $m \to T^m$. Craters and mounds correspond to peaks and valleys of permutations.

PROPOSITION 4.8. *Consider the meteor process on C_k in the stationary regime. Let $B^k_{i,j}$ be the event that* 2 *is a crater followed by a mound and another crater, with i and j sites, respectively, between the three distinguished sites. More precisely, for $i, j \geq 0$ and $k \geq i + j + 5$, let $B^k_{i,j}$ be the event that* 2 *and* $i + j + 4$ *are craters,* $i + 3$ *is a mound and m is neither a crater nor a mound for* $m = 3, 4, \dots, i + 2, i + 4, \dots, i + j + 3$.

(i) *We have*

$$\mathbb{P}(B^k_{i,j}) = \frac{2(i + j + 4)}{(i + j + 5)!}\left[\binom{i + j + 1}{i + 1} + (i + 1)\binom{i + j + 2}{i + 2}\right]. \tag{4.8}$$

(ii) *Recall events F^k_n from Theorem 4.1. If F^k_n holds, let R denote the position of the unique mound between craters at* 2 *and* $n + 3$. *For any $\varepsilon > 0$ there exist constants $c_1, c_2 > 0$ such that for $n \geq 1$ and $k \geq n + 4$,*

$$\mathbb{P}\big(|R/n - 1/2| > \varepsilon | F^k_n\big) < c_1 e^{-c_2 n}. \tag{4.9}$$

PROOF. (i) This part follows from Proposition 23 of [3]. The argument is totally analogous to that in the proof of Theorem 4.1, so we leave the details to the reader. We just note that one should take $m = i + 3$ and $n = i + j + 5$ in Proposition 23 of [3].

(ii) The function $H(x) := -x \log x - (1 - x) \log(1 - x)$ is smooth on $(0, 1)$. It is elementary to check that it is increasing on $(0, 1/2)$ and decreasing on $(1/2, 1)$. Hence, for some $c_3, c_4 > 0$ and all $x \in (0, 1)$,

$$H(x) \leq H(1/2) - c_3|x - 1/2|^2 \leq \log\left(\frac{2}{1 + c_4|x - 1/2|^2}\right). \tag{4.10}$$

By the Stirling approximation, for any $c_5 < 1 < c_6$, some m_1 and all $m \geq m_1$, we have $c_5 m \log m < \log(m!) < c_6 m \log m$. Fix any $\varepsilon > 0$, and let $c_7 > 1$ be so small that $c_7 \log(2/(1 + c_4\varepsilon^2)) := c_8 < \log 2$. For some m_1 and r_1, all $m \geq m_1$ and $r \geq r_1$ such that $m - r \geq r_1$, we have

$$\begin{aligned}\log\binom{m}{r} &= \log\left(\frac{m!}{r!(m - r)!}\right) \leq c_7\big(m \log m - r \log r - (m - r)\log(m - r)\big)\\ &= c_7 m\left(-\frac{r}{m}\log\frac{r}{m} - \left(1 - \frac{r}{m}\right)\log\left(1 - \frac{r}{m}\right)\right).\end{aligned}$$

This and (4.10) imply that if $m \geq m_1$, $r \geq r_1$, $m - r \geq r_1$, $\varepsilon > 0$ and $|r/m - 1/2| > \varepsilon$,

$$\log\binom{m}{r} \leq c_7 m \log\left(\frac{2}{1 + c_4|r/m - 1/2|^2}\right) \leq c_7 m \log\left(\frac{2}{1 + c_4\varepsilon^2}\right) = c_8 m.$$

If we take $m = i + j$, $r = i$ and we assume that $|i - (i + j)/2| > \varepsilon(i + j)/2$, then the last estimate yields for $i + j \geq m_1$ and $i, j \geq r_1$,

$$\log\binom{i + j}{i} \leq c_8(i + j),$$

and, therefore,

$$\binom{i+j}{i} \le e^{c_8(i+j)}. \tag{4.11}$$

Note that for some polynomial q_1,

$$\frac{2(i+j+4)}{(i+j+5)!}\left[\binom{i+j+1}{i+1} + (i+1)\binom{i+j+2}{i+2}\right] \le \frac{q_1(i+j)}{(i+j)!}\binom{i+j}{i}.$$

This, (4.8) and (4.11) give for i and j satisfying $|i-(i+j)/2| > \varepsilon(i+j)/2$, $i+j \ge m_1$ and $i, j \ge r_1$,

$$\mathbb{P}(B_{i,j}^k) \le \frac{q_1(i+j)e^{c_8(i+j)}}{(i+j)!}. \tag{4.12}$$

By changing the polynomial q_1, if necessary, we can drop the assumptions that $i+j \ge m_1$ and $i, j \ge r_1$.

Let

$$\Lambda(n,\varepsilon) = \{(i,j) \in \mathbb{Z} : i, j \ge 0, i+j+1 = n, |i-(i+j)/2| > \varepsilon(i+j)/2\}.$$

Recall that $c_8 < \log 2$. We obtain from (4.2) and (4.12) that for some $c_1, c_2 > 0$,

$$\begin{aligned}
\mathbb{P}(|R/n - 1/2| > \varepsilon | F_n^k) &= (\mathbb{P}(F_n^k))^{-1} \sum_{(i,j)\in\Lambda(n,\varepsilon)} \mathbb{P}(B_{i,j}^k) \\
&\le \frac{(n+4)!}{n(n+3)2^{n+1}} \sum_{(i,j)\in\Lambda(n,\varepsilon)} \frac{q_1(i+j)e^{c_8(i+j)}}{(i+j)!} \\
&= \frac{(n+4)!}{n(n+3)2^{n+1}} \sum_{(i,j)\in\Lambda(n,\varepsilon)} \frac{q_1(n-1)e^{c_8(n-1)}}{(n-1)!} \\
&\le \frac{(n+4)!}{n(n+3)2^{n+1}} n \frac{q_1(n-1)e^{c_8(n-1)}}{(n-1)!} \\
&\le c_1 e^{-c_2 n}.
\end{aligned}$$

This completes the proof. □

The results and remarks presented so far in this section indicate clearly that the crater process does not behave like a Poisson point process on C_k. There are many ways to make this intuition precise. Our next result shows that if there are very few craters, then their positions are not approximately distributed as independent uniform random variables on C_k, unlike in the case of a Poisson point process. We will prove that craters have a tendency to repel each other. This "repulsion" phenomenon is known in some other contexts; for example, it applies to eigenvalues of random matrices [17] and other determinantal processes [23].

THEOREM 4.9. *Consider the meteor process on a circular graph* C_k *with* $k \geq 3$, *and assume that the mass process* $\{\mathcal{M}_t, t \in \mathbb{R}\}$ *is in the stationary regime. Let* $\mathcal{G}_1$ *be the family of adjacent craters*, *that is*, $(i, j) \in \mathcal{G}_1$ *if an only if there are craters at* i *and* j, *and there are no craters between* i *and* j. *We define* $\mathcal{G}_2$ *as the family of pairs* (i, j) *such that there is a crater at* i *and a mound at* j, *or there is a mound at* i *and a crater at* j, *and there are neither craters nor mounds between* i *and* j. *For* $r > 2$, *let*

$$A_r^1 = \left\{ \frac{\max_{(i,j)\in\mathcal{G}_1} |i-j|}{\min_{(i,j)\in\mathcal{G}_1} |i-j|} \leq r \right\},$$

$$A_r^2 = \left\{ \frac{\max_{(i,j)\in\mathcal{G}_2} |i-j|}{\min_{(i,j)\in\mathcal{G}_2} |i-j|} \leq r \right\}.$$

(i) *Let* H_n^1 *be the event that there are exactly* n *craters at time* 0. *For every* $n \geq 2$, $p < 1$ *and* $r > 2$ *there exists* $k_1 < \infty$ *such that for all* $k \geq k_1$, $\mathbb{P}(A_r^1 | H_n^1) > p$.

(ii) *Let* H_n^2 *be the event that there are exactly* n *craters and mounds at time* 0. *For every* $n \geq 2$, $p < 1$ *and* $r > 2$ *there exists* $k_1 < \infty$ *such that for all* $k \geq k_1$, *we have* $\mathbb{P}(A_r^2 | H_n^2) > p$.

REMARK 4.10. A combinatorial result in [18], Theorem 6.1, shows that, assuming that there are exactly n craters, their most likely configuration makes them equidistant from each other. See also [25] for a closely related result. These results are not equivalent to Theorem 4.9 because the probability that one of the most likely configurations will occur does not have to be high.

PROOF OF THEOREM 4.9. Recall that, as we move around the graph C_k, we will encounter an alternating sequence of craters and mounds, separated by stretches of sites that are neither. Hence it is easy to see that part (ii) implies (i). It remains to prove (ii).

Let $G_{i,j}$ be the event that there are craters at sites i and j, and there are no craters between these two sites. Given this event, let $R_{i,j}$ be the distance from the unique mound between i and j to the closest of these vertices. We define $\widehat{G}_{i,j}$ and $\widehat{R}_{i,j}$ in an analogous way, reversing the roles of craters and mounds.

Fix an $n \geq 2$. It is elementary to see that for every $r > 2$ there exist $\varepsilon > 0$ and $c_1 > 0$ such that if A_r^2 fails to hold, then the following event must occur:

$$\begin{aligned} &\bigcup_{\substack{i,j\in V\\ |i-j|>c_1k+1}} \bigl(G_{i,j} \cap \bigl\{\bigl|R_{i,j}/|i-j-1| - 1/2\bigr| > \varepsilon\bigr\}\bigr) \\ &\quad\cup \bigcup_{\substack{i,j\in V\\ |i-j|>c_1k+1}} \bigl(\widehat{G}_{i,j} \cap \bigl\{\bigl|\widehat{R}_{i,j}/|i-j-1| - 1/2\bigr| > \varepsilon\bigr\}\bigr). \end{aligned} \tag{4.13}$$

If $G_{i,j}$ holds, then the value of $R_{i,j}$ does not depend on the positions of craters and mounds outside the interval between i and j. Hence

$$\begin{aligned}
&\mathbb{P}(G_{i,j} \cap \{|R_{i,j}/|i-j-1| - 1/2| > \varepsilon\} | H_n^2) \\
&\quad = \mathbb{P}(G_{i,j}|H_n^2)\mathbb{P}(|R_{i,j}/|i-j-1| - 1/2| > \varepsilon | G_{i,j} \cap H_n^2) \\
&\quad \le \mathbb{P}(|R_{i,j}/|i-j-1| - 1/2| > \varepsilon | G_{i,j} \cap H_n^2) \\
&\quad = \mathbb{P}(|R_{i,j}/|i-j-1| - 1/2| > \varepsilon | G_{i,j}).
\end{aligned}$$

Proposition 4.8(ii) yields for some $c_2, c_3 > 0$ and i and j such that $|i-j| > c_1 k + 1$,

$$\begin{aligned}
&\mathbb{P}(G_{i,j} \cap \{|R_{i,j}/|i-j-1| - 1/2| > \varepsilon\} | H_n^2) \\
&\quad \le \mathbb{P}(|R_{i,j}/|i-j-1| - 1/2| > \varepsilon | G_{i,j}) \\
&\quad \le c_2 e^{-c_3|i-j-1|} \le c_2 e^{-c_3 c_1 k}.
\end{aligned} \tag{4.14}$$

Interchanging the roles of craters and mounds, we obtain for i and j such that $|i-j| > c_1 k + 1$,

$$\mathbb{P}(\widehat{G}_{i,j} \cap \{|\widehat{R}_{i,j}/|i-j-1| - 1/2| > \varepsilon\} | H_n^2) \le c_2 e^{-c_3 c_1 k}.$$

This, (4.14) and (4.13) imply that

$$\mathbb{P}((A_r^2)^c | H_n^2) \le 2 \sum_{\substack{i,j \in V \\ |i-j| > c_1 k + 1}} c_2 e^{-c_3 c_1 k} \le 2k^2 c_2 e^{-c_3 c_1 k}.$$

The last quantity goes to 0 as $k \to \infty$. This completes the proof. $\square$

The last question that we are going to address in this section concerns the age of the oldest exposed soil. A meteor hit displaces some soil, and we can imagine that the displaced soil is placed on the top of the soil already present at the site where it is deposited. Hence the age of the oldest exposed soil is the minimum over all n of $\widetilde{T}^n := \max(T^{n-1}, T^n, T^{n+1})$.

THEOREM 4.11. *For any $\varepsilon > 0$ and $p < 1$ there exists k_1 such that for $k \ge k_1$,*

$$\mathbb{P}\Big(\Big|\min_{1 \le n \le k} \widetilde{T}^n - (1/3)\log k\Big| < \varepsilon \log k\Big) > p.$$

PROOF. Consider any $\alpha \in (0, 2/3)$, and let $\beta = 2/3 - \alpha > 0$. The probability that T^n is among the k^α lowest values of $\{T^j, 1 \le j \le k\}$ is less than $2k^\alpha/k = 2k^{\alpha-1}$. Hence, for a fixed n and large k, the probability that $\widetilde{T}^n$ is among the k^α lowest values of $\{T^j, 1 \le j \le k\}$ is less than $2(2k^{\alpha-1})^3 = 16k^{3(\alpha-1)} = 16k^{-1-3\beta}$ (the dependence between the relevant events is negligible for large k). It follows that the probability that there exists a site n such that $\widetilde{T}^n$ is among the k^α lowest values of $\{T^j, 1 \le j \le k\}$ is less than $k16k^{-1-3\beta} = 16k^{-3\beta}$. The last quantity goes to 0 as $k \to \infty$.

Consider any $\gamma \in (2/3, 1)$ and let $\lambda = \gamma - 2/3 > 0$. The probability that T^n is among the k^γ lowest values of $\{T^j, 1 \le j \le k\}$ is more than $k^\gamma/(2k) = (1/2)k^{\gamma-1}$. Hence, for a fixed n and large k, the probability that $\widetilde{T}^n$ is among the k^γ lowest values of $\{T^j, 1 \le j \le k\}$ is more than $(1/2)((1/2)k^{\gamma-1})^3 = (1/16)k^{3(\gamma-1)} = (1/16)k^{-1+3\lambda}$. It follows that the probability that there exists a site n such that $1 \le n = 3i \le k$, $i \in \mathbb{Z}$ and $\widetilde{T}^n$ is among the k^γ lowest values of $\{T^j, 1 \le j \le k\}$ is more than $1 - (1 - (1/16)k^{-1+3\lambda})^{k/6}$. The last quantity goes to 1 as $k \to \infty$.

Let J be the rank of $\min_{1 \le j \le k} \widetilde{T}^j$ among the ordered values of $\{T^j, 1 \le j \le k\}$. We have shown that for any $0 < \alpha < 2/3 < \gamma < 1$, we have

$$\lim_{k\to\infty} \mathbb{P}\big(k^\alpha < J < k^\gamma\big) = 1. \tag{4.15}$$

Note that $\{-T^j, 1 \le j \le k\}$ are i.i.d., with the exponential distribution with mean 1. Let $Y_{(n)}$ denote the nth order statistic for $\{-T^j, 1 \le j \le k\}$. It follows from [15], Theorem 2.2.1, that for any fixed $a \in (0, 1)$, random variables $k^{a/2}(Y_{(k-k^a)} - (1-a)\log k)$ converge weakly to the standard normal random variable as $k \to \infty$. This and (4.15) easily imply the theorem. $\square$

5. Mass distribution. Section 4 was concerned with the distribution of craters, that is, sites where the mass M^j is 0. This section will present some results on the mass distribution at all sites. In other words, we will consider the nondegenerate part of the mass distribution at a site.

THEOREM 5.1. *Suppose that $d \ge 1$, and let $\{\mathcal{M}_t, t \ge 0\} = \{(M_t^1, M_t^2, \dots, M_t^k), t \ge 0\}$ be the mass process on $G = C_n^d$ (the product of d copies of the cycle C_n), under the stationary measure Q_k (here $k = nd$). Assume that $\sum_{x\in V} M_0^x = k$ under Q_k. We have*

$$\mathbb{E}_{Q_k} M_0^x = 1, \qquad x \in V, \tag{5.1}$$

$$\lim_{k\to\infty} \mathrm{Var}_{Q_k} M_0^x = 1, \qquad x \in V, \tag{5.2}$$

$$\lim_{k\to\infty} \mathrm{Cov}_{Q_k}\big(M_0^x, M_0^y\big) = -\frac{1}{2d}, \qquad x \leftrightarrow y, \tag{5.3}$$

$$\lim_{k\to\infty} \mathrm{Cov}_{Q_k}\big(M_0^x, M_0^y\big) = 0, \qquad x \ne y \text{ and } x \not\leftrightarrow y. \tag{5.4}$$

PROOF. By symmetry, $\mathbb{E}_{Q_k} M_0^x = \mathbb{E}_{Q_k} M_0^y$ for all $x, y \in V$. Since $\sum_{x\in V} M_0^x = k$ under Q_k, we must have $\mathbb{E}_{Q_k} M_0^x = 1$ for $x \in V$. This proves (5.1).

We will base our estimates for $\mathrm{Var}_{Q_k} M_0^x$ and $\mathrm{Cov}_{Q_k}(M_0^x, M_0^y)$ on a representation of M_0^x using WIMPs. Let Z and $\widetilde{Z}$ be defined as Z^1 and Z^2 in Definition 2.1. In particular, $\mathbb{P}(Z_0 = x) = \mathbb{P}(\widetilde{Z}_0 = x) = M_0^x/k$ for $x \in V$.

Note that since the state space C_k^{2d} for the process $(Z, \widetilde{Z})$ is finite, the process has a stationary distribution. The stationary distribution is unique because all states communicate. We will estimate the probability that $Z_t = \widetilde{Z}_t$ under the stationary distribution. Let $\overline{Z}_t = Z_t - \widetilde{Z}_t$. It is easy to see that $\overline{Z}_t$ is a Markov process (although a function of a Markov process is not necessarily Markov). The state space for $\overline{Z}_t$ may be identified with V in the obvious way. Let $\{\pi_x, x \in V\}$ be the set of stationary probabilities for the discrete time Markov chain Z_j^* embedded in $\overline{Z}_t$.

First, we will discuss the case $d = 1$. We claim that, in this case, for some $c_1 > 0$, $\pi_1 = \pi_{n-1} = c_1/2$ and $\pi_j = c_1$ for $j \neq 1, n-1$. It is easy to check that the following equations define the stationary probabilities, and these equations are satisfied by the probabilities specified above:

$$\begin{aligned}
&\pi_0 = \tfrac{1}{2}\pi_0 + \tfrac{1}{2}\pi_1 + \tfrac{1}{2}\pi_{n-1}, \qquad \pi_1 = \tfrac{1}{2}\pi_2, \qquad \pi_{n-1} = \tfrac{1}{2}\pi_{n-2},\\
&\pi_2 = \tfrac{1}{2}\pi_3 + \tfrac{1}{2}\pi_1 + \tfrac{1}{4}\pi_0, \qquad \pi_{n-2} = \tfrac{1}{2}\pi_{n-3} + \tfrac{1}{2}\pi_{n-1} + \tfrac{1}{4}\pi_0,\\
&\pi_j = \tfrac{1}{2}\pi_{j-1} + \tfrac{1}{2}\pi_{j+1}, \qquad j \neq n-2, n-1, 0, 1, 2.
\end{aligned}$$

Of course, c_1 is chosen so that $\sum_n \pi_n = 1$. The mean holding time for $\overline{Z}_t$ is 1 in the state 0 and it is $1/2$ in all other states. This and the formulas for π_j's imply that

$$\begin{aligned}
&\lim_{k\to\infty} (k/2)\mathbb{E}_{Q_k}(Z_t = \widetilde{Z}_t) = \lim_{k\to\infty} (k/2)\mathbb{E}_{Q_k}(\overline{Z}_t = 0) = 1,\\
&\lim_{k\to\infty} 2k\mathbb{E}_{Q_k}(Z_t - \widetilde{Z}_t = 1) = \lim_{k\to\infty} 2k\mathbb{E}_{Q_k}(Z_t - \widetilde{Z}_t = -1) = 1,\\
&\lim_{k\to\infty} k\mathbb{E}_{Q_k}(Z_t - \widetilde{Z}_t = j) = 1, \qquad j \neq -1, 0, 1.
\end{aligned} \tag{5.5}$$

The case $d \geq 2$ is similar but requires different notation. Recall that $\mathbf{0} = (0, \dots, 0)$. Let $\mathbf{a}$ be set of all vertices $(a_1, \dots, a_d)$ such that $|a_i| = |a_j| = 1$ for some i and j, and $a_m = 0$ for all $m \neq i, j$. Let $\mathbf{b}$ be set of all vertices $(b_1, \dots, b_d)$ such that $|b_i| = 2$ for some i, and $b_m = 0$ for all $m \neq i$. Let $\mathbf{h}$ be set of all vertices $(h_1, \dots, h_d)$ such that $|h_i| = 1$ for some i, and $h_m = 0$ for all $m \neq i$. Let $\mathbf{g} = V \setminus (\{\mathbf{0}\} \cup \mathbf{a} \cup \mathbf{b} \cup \mathbf{h})$.

We claim that for some $c_1 > 0$, $\pi_x = (1 - \frac{1}{2d})c_1$ for all $x \in \mathbf{h}$ and $\pi_x = c_1$ for all other $x \in V$. It is easy to check that the following equations define the stationary probabilities, and these equations are satisfied by the probabilities specified above:

$$\begin{aligned}
&\pi_{\mathbf{0}} = \frac{1}{2d}\pi_{\mathbf{0}} + 2d\frac{1}{2d}\pi_x, \qquad x \in \mathbf{h},\\
&\pi_x = \frac{1}{2d}\pi_y + (2d-2)\frac{1}{2d}\pi_z, \qquad x \in \mathbf{h}, y \in \mathbf{b}, z \in \mathbf{a},\\
&\pi_x = 2\left(\frac{1}{2d}\right)^2 \pi_{\mathbf{0}} + 2\frac{1}{2d}\pi_y + (2d-2)\frac{1}{2d}\pi_z, \qquad x \in \mathbf{a}, y \in \mathbf{h}, z \in \mathbf{g},
\end{aligned}$$

$$\pi_x = \left(\frac{1}{2d}\right)^2 \pi_{\mathbf{0}} + \frac{1}{2d}\pi_y + (2d-1)\frac{1}{2d}\pi_z, \qquad x \in \mathbf{b}, y \in \mathbf{h}, z \in \mathbf{g},$$

$$\pi_x = 2d\frac{1}{2d}\pi_y, \qquad x \in \mathbf{g}, y \in \mathbf{a} \cup \mathbf{b} \cup \mathbf{g}.$$

Recall that c_1 is chosen so that $\sum_n \pi_n = 1$, the mean holding time for $\overline{Z}_t$ is 1 in the state 0 and it is $1/2$ in all other states. This and the formulas for π_j's imply that

$$\lim_{k\to\infty}(k/2)\mathbb{E}_{Q_k}(Z_t = \widetilde{Z}_t) = \lim_{k\to\infty}(k/2)\mathbb{E}_{Q_k}(\overline{Z}_t = \mathbf{0}) = 1,$$

$$\lim_{k\to\infty}\frac{2d}{2d-1}k\mathbb{E}_{Q_k}(Z_t - \widetilde{Z}_t = x) = 1, \qquad x \in \mathbf{h}, \tag{5.6}$$

$$\lim_{k\to\infty}k\mathbb{E}_{Q_k}(Z_t - \widetilde{Z}_t = x) = 1, \qquad x \notin \{\mathbf{0}\} \cup \mathbf{h}.$$

Let $\alpha_{\mathbf{0}} = 2$, $\alpha_x = 1 - \frac{1}{2d}$ for $x \in \mathbf{h}$ and $\alpha_x = 1$ for all other x. By (5.5) and (5.6), for any fixed $x \in V$ and an arbitrarily small $\varepsilon > 0$, there exists k_1 so large that for any $k \geq k_1$, the probability that $Z_t - \widetilde{Z}_t = x$ under the stationary distribution is in the interval $((1-\varepsilon)\alpha_x/k, (1+\varepsilon)\alpha_x/k)$. Hence, for $y \in V$,

$$\mathbb{P}_{Q_k}(Z_0 = y, \widetilde{Z}_0 = y + x) \in \big((1-\varepsilon)\alpha_x/k^2, (1+\varepsilon)\alpha_x/k^2\big). \tag{5.7}$$

Let $\mathcal{G}_t = \sigma(\mathcal{M}_s, 0 \leq s \leq t)$. It is easy to see that

$$\mathbb{P}_{Q_k}(Z_0 = x | \mathcal{G}_0) = M_0^x/k.$$

The random variables Z_0 and $\widetilde{Z}_0$ are conditionally independent given $\mathcal{G}_0$, so

$$\mathbb{P}_{Q_k}(Z_0 = y, \widetilde{Z}_0 = y + x | \mathcal{G}_0) = M_0^y M_0^{y+x}/k^2.$$

Thus

$$\begin{aligned}\mathbb{E}_{Q_k}\big(M_0^y M_0^{y+x}\big) &= k^2\mathbb{E}_{Q_k}\mathbb{P}_{Q_k}(Z_0 = y, \widetilde{Z}_0 = y + x | \mathcal{G}_0)\\ &= k^2\mathbb{P}_{Q_k}(Z_0 = y, \widetilde{Z}_0 = y + x).\end{aligned}$$

This and (5.7) yield, for $k > k_1$,

$$(1-\varepsilon)\alpha_x \leq \mathbb{E}_{Q_k}\big(M_0^y M_0^{y+x}\big) \leq (1+\varepsilon)\alpha_x.$$

Since $\varepsilon > 0$ is arbitrarily small, it follows that

$$\lim_{k\to\infty}\mathbb{E}_{Q_k}\big(M_0^y M_0^{y+x}\big) = \alpha_x.$$

For $x = \mathbf{0}$, we obtain $\lim_{k\to\infty}\mathbb{E}_{Q_k}(M_0^y)^2 = 2$. This and (5.1) imply that

$$\lim_{k\to\infty}\operatorname{Var}_{Q_k} M_0^y = 1.$$

For $x \in \mathbf{h}$, we have $\lim_{k\to\infty}\mathbb{E}_{Q_k}(M_0^y M_0^{y+x}) = 1 - \frac{1}{2d}$, so, in view of (5.1),

$$\lim_{k\to\infty}\operatorname{Cov}_{Q_k}\big(M_0^y, M_0^{y+x}\big) = -\frac{1}{2d}.$$

Finally, for $x \notin \{\mathbf{0}\} \cup \mathbf{h}$, we have $\lim_{k\to\infty} \mathbb{E}_{Q_k}(M_0^y M_0^{y+x}) = 1$, and, therefore,

$$\lim_{k\to\infty} \mathrm{Cov}_{Q_k}(M_0^y, M_0^{y+x}) = 0.$$

This completes the proof. $\square$

REMARK 5.2. (i) It has been shown in [6] (a follow-up paper) that the distributions of M_0^1 under the stationary measures Q_k converge as $k \to \infty$. We have neither explicit description nor detailed information about the limit distribution. We performed a number of long simulations. Figure 1 illustrates some of the numerical results. The figure on the left shows the empirical distribution of masses $\{M^j_{10{,}000{,}000}, 1 \le j \le 60{,}000\}$, based on a single simulation with ten million jumps ("meteor hits") for a circular graph $C_{60{,}000}$. The distribution has an atom at 0 of size about $1/3$, as predicted by Theorem 4.1. The distribution does not appear to have any other atoms. The graph on the right shows the "Q–Q" plot (quantile on quantile plot) for the continuous component of the empirical distribution of masses versus the best matching gamma density (in the sense of matching the first two moments), for a simulation on the graph C_{6000}. The "Q–Q" plot shows convincingly that the distribution is not in the gamma family. We will return to this point in part (iii) of this remark.

(ii) An argument similar to that in the proof of Theorem 5.1 leads to a (nonasymptotic) formula for the third moment of M_0^1, for a fixed circular graph C_k. The calculation is based on the derivation of the stationary distribution for the Markov process consisting of three dependent continuous time random walks. The stationary distribution can be explicitly calculated using computer algebra for low values of k. The values of the third moment of M_0^1 seem to converge to 4.75531 as k goes to infinity. This value is consistent with the results of computer simulations.

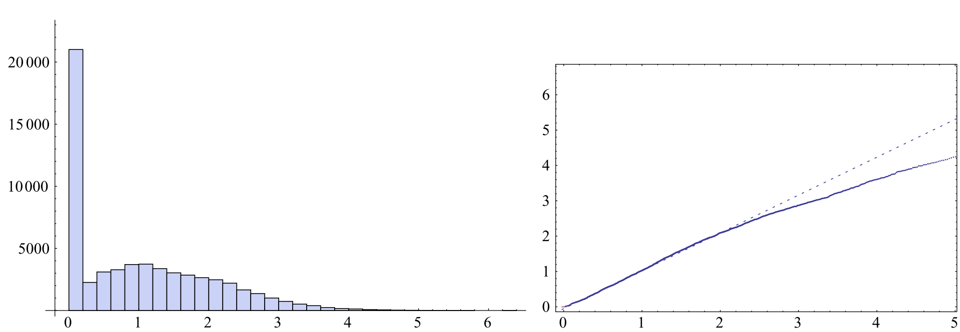

FIG. 1. *The figure on the left shows the empirical distribution of masses* $\{M^j_{10{,}000{,}000}, 1 \le j \le 60{,}000\}$, *based on a single simulation with ten million jumps* (*"meteor hits"*) *for a circular graph* $C_{60{,}000}$. *The distribution has an atom at* 0 *of* (*theoretical*) *size* $1/3$. *The graph on the right shows the "Q–Q" plot* (*quantile on quantile plot*) *for the continuous component of the empirical distribution of masses versus the best matching gamma density* (*in the sense of matching the first two moments*), *for a simulation on the graph* C_{6000}.

Calculating the stationary distribution for the Markov chain of three random walks quickly becomes a time consuming task because the state space of the Markov chain has k^3 elements, assuming that the cycle has size k. To reduce the size of the state space, we collapsed the states that were images of each other under symmetries of the cycle. For example, for $k = 20$, the state space size was reduced from $20^3 = 8000$ to 44.

(iii) It follows from our estimates that the limiting distribution of the mass size at a given point, after removing the atom at 0, does not belong to the gamma family. For a gamma random variable X with density $x^{\alpha-1}\exp(-x/\beta)/(\Gamma(\alpha)\beta^\alpha)$, we have $EX^j = \beta^j\alpha(\alpha+1)\cdots(\alpha+j-1)$. In particular, $EX = \beta\alpha$, $EX^2 = \beta^2\alpha(\alpha+1)$ and $EX^3 = \beta^3\alpha(\alpha+1)(\alpha+2)$. Let W be M_0^1 conditioned to be nonzero. In our case, under the stationary distribution Q_k, we have $EW = 3/2$, $EW^2 = 3$ and $EW^3 \approx 4.755$. If we have $EX = 3/2$ and $EX^2 = 3$ for a gamma distribution, then $EX^3 = 7.5 \neq 4.755$. There are no values of α and β that would make the moments of W match the moments of a gamma distribution even in an approximate sense.

(iv) Numerical calculations suggest that $(M_0^1)^2$ and M_0^j are asymptotically correlated, when $k \to \infty$. Hence, it appears that M_0^1 and M_0^j are asymptotically dependent, when $k \to \infty$. We do not have a heuristic explanation for the lack of asymptotic correlation of M_0^1 and M_0^j, for $j \geq 3$, proved in Theorem 5.1.

(v) When $k = 2$ or 3, we can provide an explicit description of the stationary distribution for the mass process $\mathcal{M}_t$ on the circular graph C_k. If $k = 2$, then the stationary distribution of $\mathcal{M}_t$ has two atoms of size 1/2. One atom is the measure that gives mass 2 to site 1 and mass 0 to site 2. The other atom is the measure that gives mass 2 to site 2 and mass 0 to site 1.

Suppose that $k = 3$ and for $j = 1, 2, 3$, let μ_j be the random measure which gives mass 0 to site j, $\mu_j(j+1)$ is the uniform random variable on $[0, 2]$ and $\mu_j(j+1) = 2 - \mu_j(j+2)$. Then the stationary distribution for $\mathcal{M}_t$ is the mixture, with equal weights, of μ_j, $j = 1, 2, 3$. It is an elementary exercise to check that the given measures are stationary.

(vi) Consider the meteor process on a circular graph C_k, and let $M_t^{1,n} = \sum_{j=1}^n M_t^j$. Then Theorem 5.1 implies that for any fixed n, $\lim_{k\to\infty} \operatorname{Var}_{Q_k} M_0^{1,n} = 1$. In other words, although the expected mass in an interval of length n, that is, $\mathbb{E}_{Q_k} M_0^{1,n} = n$, grows with n, the variance of this mass does not grow (in the limit when $k \to \infty$).

More generally, consider the meteor process on the product C_n^d of circular graphs, and let $k = dn$. For a set $A \subset V$, let $M_t^A = \sum_{x\in A} M_t^x$. Let ∂A be the number of edges joining two vertices of which exactly one is in A. Then Theorem 5.1 implies that for any fixed A, $\lim_{k\to\infty} \operatorname{Var}_{Q_k} M_0^A = |\partial A|/(2d)$. Obviously, $\mathbb{E}_{Q_k} M_0^A = |A|$. The mass enclosed in each of the shapes in Figure 2 has the same (asymptotic) variance.

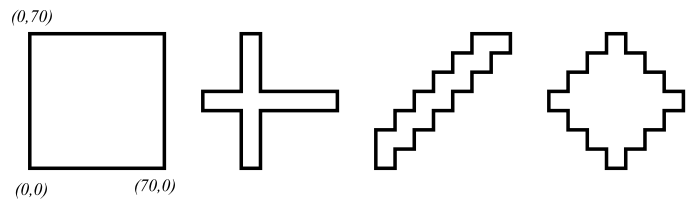

FIG. 2. *All curves have the same height and width. They all have the same "boundary length" $|\partial A|$, where A denotes the set of vertices inside the given closed curve. The asymptotic variance of the mass enclosed by one of these four curves has the same value as for any other of these curves.*

Consider the meteor process on a circular graph C_k with $k \geq 4$ and assume that the mass process $\{\mathcal{M}_t, t \in \mathbb{R}\}$ is in the stationary regime. We will estimate the expected value of the height of a crater rim, that is, the expected value of the mass at a site that is adjacent to a crater. Note that the expected value of the mass at a uniformly chosen noncrater is $3/2$ because the expected value of the mass at a site is 1 and the probability that a site is a crater is $1/3$.

PROPOSITION 5.3. *Consider the meteor process on a circular graph C_k with $k \geq 6$, and assume that the mass process $\{\mathcal{M}_t, t \in \mathbb{R}\}$ is in the stationary regime. Then*

$$1.625 = 13/8 < \mathbb{E}_Q\bigl(M_0^2 | M_0^1 = 0\bigr) < 5/3 \approx 1.667. \tag{5.8}$$

PROOF. Recall that T^j denotes the time of the last jump of N^j before 0, that is, $T^j = \sup\{t \leq 0 : N_t^j \neq N_{t-}^j\}$. The event $A := \{M_0^1 = 0\}$ is equivalent to $\{T^2 < T^1 > T^k\}$. It is easy to see that the conditional distribution of T^1 given $\{T^2 < T^1 > T^k\}$ is the same as the distribution of $\max(T^k, T^1, T^2)$. The density of $-\max(T^k, T^1, T^2)$ is $3e^{-3t}$.

The conditional distribution of $\mathcal{M}_{t-}$ given $A \cap \{T^1 = t\}$ is the stationary distribution because the event $A \cap \{T^1 = t\}$ is determined by $\{N_s^j, t \leq s \leq 0\}$, and the value of $\mathcal{M}_{t-}$ is determined by $\{N_s^j, s < t\}$.

Fix some $S > 0$, and assume that $A \cap \{T^1 = -S\}$ occurred. For $t \in [0, S]$, let $F_t^j = \mathbb{E}(M_{-S+t}^j | A \cap \{T^1 = -S\})$, and note that $F_0^1 = 0$, $F_0^2 = F_0^k = 3/2$ and $F_0^j = 1$ for all other j. Given $A \cap \{T^1 = -S\}$, meteors hit sites $3, 4, \dots, k-1$ at a constant rate of 1 hit per unit of time during the time interval $(-S, 0)$, so for $t \in [0, S]$,

$$\begin{aligned}
&\frac{d}{dt} F_t^2 = \frac{1}{2} F_t^3, \\
&\frac{d}{dt} F_t^3 = \frac{1}{2} F_t^4 - F_t^3, \\
&\frac{d}{dt} F_t^j = \frac{1}{2} F_t^{j-1} + \frac{1}{2} F_t^{j+1} - F_t^j, \qquad j = 4, \dots, k-2.
\end{aligned} \tag{5.9}$$

These equations and the initial conditions imply that $F_t^3 > e^{-t}$ and, therefore, $F_t^2 > 3/2 + (1 - e^{-t})/2$ for $t \in (0, S]$. It follows that

$$\mathbb{E}_Q(M_0^2 | M_0^1 = 0) > \int_0^\infty (3/2 + (1 - e^{-s})/2) 3e^{-3s}\, ds = 13/8.$$

We also have $F_t^3 < 1$ and, therefore, $F_t^2 < 3/2 + t/2$ for $t \in (0, S]$. Hence

$$\mathbb{E}_Q(M_0^2 | M_0^1 = 0) < \int_0^\infty (3/2 + s/2) 3e^{-3s}\, ds = 5/3.$$

This completes the proof. $\square$

REMARK 5.4. Computer simulations show that $\mathbb{E}_Q(M_0^2 | M_0^1 = 0) \approx 1.6443$.

We note that one can derive sharper estimates for F_t^2 using (5.9) and hence sharper estimates in (5.8).

6. Meteor processes on noncircular graphs.

Consider a circular graph C_k, and suppose that the total mass $\sum_{j=1}^k M_0^j$ is equal to k. Then it is obvious that $\mathbb{E}_{Q_k} M_0^1 = 1$ for every k, by symmetry. However, the fact that $\lim_{k \to \infty} \mathrm{Var}_{Q_k} M_0^1 = 1$, proved in Theorem 5.1, does not seem to be obvious. We will show that under some structural assumptions on the graph G, the variance of M_0^x under the stationary distribution cannot be too large. We will show that the bound for the variance of M_0^x depends mainly on the degree of the vertex.

A graph is called distance-transitive if for any two vertices v and w at any distance i, and any other two vertices x and y at the same distance, there is an automorphism of the graph that carries v to x and w to y.

THEOREM 6.1. *Assume that G is a distance-transitive ρ-regular graph. Assume that $\sum_{x \in V} M_0^x = |V|$. Then under the stationary distribution, for any $x \in V$,*

$$\mathrm{Var}_Q M_0^x \le \frac{\rho + 1}{\rho - 1 + 2\rho/(|V| - 1)}.$$

PROOF. By symmetry, $\mathbb{E}_Q M_0^x = 1$, for all $x \in V$.

Let Z and $\widetilde{Z}$ be defined as Z^1 and Z^2 in Definition 2.1. In particular, $\mathbb{P}(Z_0 = x) = \mathbb{P}(\widetilde{Z}_0 = x) = M_0^x / |V|$ for $x \in V$.

Note that since the state space V^2 for the process $(Z, \widetilde{Z})$ is finite, the process has a stationary distribution. The stationary distribution is unique because all states communicate. We will estimate the probability that $Z_t = \widetilde{Z}_t$ under the stationary distribution.

Fix any vertex and label it $\mathbf{0}$. Let Z^1 be a continuous time Markov process on V defined as follows. We let Z_0^1 be a vertex uniformly chosen from all vertices x with the property that the distance from x to $\mathbf{0}$ is the same as the distance from

Z_0 to $\widetilde{Z}_0$. The process Z^1 jumps if an only if $(Z, \widetilde{Z})$ jumps. At a time t of a jump of $(Z, \widetilde{Z})$, the process Z^1 jumps to one of the nearest neighbors of Z^1_{t-}, whose distance from $\mathbf{0}$ is the same as the distance between Z_t and $\widetilde{Z}_t$. The process Z^1 is a continuous time Markov process on V, with the mean holding time equal to $1/2$ at all vertices $x \neq \mathbf{0}$. The mean holding time for Z^1 at $\mathbf{0}$ is $(1 - 1/\rho)^{-1}$. If $Z^1_t = \mathbf{0}$, the next jump it will take will be to a vertex at a distance either 1 or 2 from $\mathbf{0}$. If $Z^1_t \neq \mathbf{0}$, then the next jump will be to a neighbor of Z^1_t. Let Z^2_t be a continuous time symmetric nearest neighbor random walk on V, with the mean holding time equal to $1/2$ at all vertices $x \neq \mathbf{0}$, and mean holding time at $\mathbf{0}$ equal to $(1-1/\rho)^{-1}$. The only difference between Z^1 and Z^2 is that Z^2 can jump from $\mathbf{0}$ only to a nearest neighbor while Z^1 can jump from $\mathbf{0}$ to some other vertices.

The long run proportion of time spent by Z^2 at $\mathbf{0}$ is

$$\frac{(1-1/\rho)^{-1}}{(1-1/\rho)^{-1} + (|V|-1)/2} = \frac{\rho}{\rho + (\rho-1)(|V|-1)/2}.$$

After every jump of Z^1 from $\mathbf{0}$, this process will take some time, not necessarily zero, until it hits a neighbor of $\mathbf{0}$. Hence, the long run proportion of time spent by Z^1 at $\mathbf{0}$ is less than or equal to

$$\frac{\rho}{\rho + (\rho-1)(|V|-1)/2}.$$

By symmetry, for any $x \in V$, the long run proportion of time spent by $(Z, \widetilde{Z})$ at (x, x) is less than or equal to

$$\frac{\rho}{|V|(\rho + (\rho-1)(|V|-1)/2)}.$$

Hence, for any $x \in V$,

$$\mathbb{P}_Q(Z_0 = \widetilde{Z}_0 = x) = \frac{\rho}{|V|(\rho + (\rho-1)(|V|-1)/2)}. \tag{6.1}$$

Let $\mathcal{G}_t = \sigma(\mathcal{M}_s, 0 \le s \le t)$. Then, for $x \in V$,

$$\mathbb{P}_Q(Z_0 = x | \mathcal{G}_0) = M^x_0 / |V|.$$

The random variables Z_0 and $\widetilde{Z}_0$ are conditionally independent given $\mathcal{G}_0$, so

$$\mathbb{P}_Q(Z_0 = \widetilde{Z}_0 = x | \mathcal{G}_0) = \left(M^x_0/|V|\right)^2.$$

Thus

$$\mathbb{E}_Q\left(M^x_0\right)^2 = |V|^2 \mathbb{E}_Q \mathbb{P}_Q(Z_0 = \widetilde{Z}_0 = x | \mathcal{G}_0) = |V|^2 \mathbb{P}_{Q_k}(Z_0 = \widetilde{Z}_0 = x).$$

This and (6.1) yield

$$\mathbb{E}_Q\left(M^x_0\right)^2 \le \frac{|V|^2 \rho}{|V|(\rho + (\rho-1)(|V|-1)/2)}.$$

Since $\mathbb{E}_Q M_0^x = 1$, we obtain

$$\operatorname{Var}_Q M_0^x \le \frac{|V|^2\rho}{|V|(\rho + (\rho - 1)(|V| - 1)/2)} - 1 = \frac{\rho + 1}{\rho - 1 + 2\rho/(|V| - 1)}.$$

This completes the proof. $\square$

Recall that T_t^v denotes the time of the last jump of N^v on the interval $[0, t]$, with the convention that $T_t^v = -1$ if there were no jumps on this interval.

THEOREM 6.2. *Suppose that G is a complete graph with k vertices $\{1, 2, \dots, k\}$, and recall the Poisson processes N^m. Let Q_k be the stationary distribution for the mass process $\mathcal{M}$. When $k \to \infty$, processes $\{M_t^1 - M_0^1 - t + T_t^1, t \ge 0\}$, under Q_k, converge weakly to the process identically equal to 0 in the Skorokhod space $D([0, \infty), \mathbb{R})$.*

COROLLARY 6.3. *Under assumptions of Theorem 6.2, we have the following*:

(i) *the distributions of M_0^1 under Q_k converge to the exponential distribution with mean 1, when $k \to \infty$*;

(ii) *there is propagation of chaos*; *that is*, *for any finite $n \ge 2$*, *the distributions of $\{M_t^j, t \ge 0\}$, $j = 1, \dots, n$*, *are asymptotically independent*, *when $k \to \infty$*.

PROOF OF THEOREM 6.2. Let $x^+ = \max(x, 0)$. Let $R_{s,t}^k$ be the mass moved to state 1 during the time interval $[s, t]$, that is, $R_{s,t}^k = \sum_{u \in [s,t]} (M_u^1 - M_{u-}^1)^+$. If t_1 and t_2 are any two consecutive jumps of N^1, then $M_{t_1}^1 = 0$ and $M_u^1 - M_{t_1}^1 = R_{u,t_1}^k$ for all $u \in (t_1, t_2)$. Hence it will suffice to prove that for any two fixed rational numbers $0 < t_1, t_2 < \infty$, R_{t_1,t_2}^k converges to $t_2 - t_1$ weakly, as $k \to \infty$.

Let Z^j's be defined as in Definition 2.1. Let $\mathcal{G}_t = \sigma(\mathcal{M}_s, 0 \le s \le t)$. Then for any $t \ge 0$, a.s.,

$$\mathbb{P}_{Q_k}(Z_t^j = 1 | \mathcal{G}_t) = M_t^1 / k.$$

The processes $\{Z_s^j, s \in [0, t]\}$, $j \ge 1$, are conditionally independent given $\mathcal{G}_t$, so by the law of large numbers, for every $t \ge 0$, a.s.,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^n \mathbf{1}_{\{Z_t^j = 1\}} = M_t^1 / k. \tag{6.2}$$

Since the process M^1 has only a finite number of jumps on any finite time interval, the convergence in (6.2) holds uniformly on every interval of the form $[t_1, t_2]$, with $0 < t_1 < t_2 < \infty$. Fix any $0 < t_1 < t_2 < \infty$ and let

$$A(k, n) = \frac{k}{n} \sum_{u \in [t_1, t_2]} \sum_{j=1}^n \mathbf{1}_{\{Z_u^j = 1, Z_{u-}^j \ne 1\}}.$$

In view of earlier remarks, it will suffice to prove that, in probability,

$$\lim_{k\to\infty}\lim_{n\to\infty} A(k,n) = t_2 - t_1.$$

It will be enough to show that

$$\lim_{k\to\infty}\lim_{n\to\infty} \mathbb{E}_{Q_k} A(k,n) = t_2 - t_1 \tag{6.3}$$

and

$$\lim_{k\to\infty}\lim_{n\to\infty} \operatorname{Var}_{Q_k} A(k,n) = 0. \tag{6.4}$$

Since all Z^j's have the same distribution, to prove (6.3), it will suffice to show that

$$\lim_{k\to\infty} k\mathbb{E}_{Q_k} \sum_{u\in[t_1,t_2]} \mathbf{1}_{\{Z_u^1=1, Z_{u-}^1\neq 1\}} = t_2 - t_1. \tag{6.5}$$

By symmetry, $\mathbb{P}_{Q_k}(Z_{t_1}^1 = 1) = 1/k$. After the process Z^1 jumps to some other state, it has probability less than $1 - e^{-(t_2-t_1)/(k-1)}$ of jumping to 1 in the remaining time in the interval $[t_1, t_2]$. If it jumps back to 1 and then again to another state, it has, once again, probability less than $1 - e^{-(t_2-t_1)/(k-1)}$ of jumping to 1 in the remaining time in the interval $[t_1, t_2]$. A similar argument applies to further possible jumps to 1. Hence, if we denote consecutive jumps of Z^1 to the state 1 on the interval $(t_1, t_2]$ by $S_1, S_2, \dots$, then

$$\begin{aligned}\mathbb{P}_{Q_k}\big(Z_{t_1}^1 = 1, S_m \le t_2\big) &\le \frac{1}{k}\big(1 - e^{-(t_2-t_1)/(k-1)}\big)^m \\ &\le (t_2 - t_1)^m (k-1)^{-m-1},\end{aligned} \tag{6.6}$$

and, therefore,

$$\mathbb{E}_{Q_k} \sum_{u\in(t_1,t_2]} \mathbf{1}_{\{Z_{t_1}^1=1\}} \mathbf{1}_{\{Z_u^1=1, Z_{u-}^1\neq 1\}} \le \sum_{m\ge 1} (t_2 - t_1)^m (k-1)^{-m-1}.$$

This implies that

$$\lim_{k\to\infty} k\mathbb{E}_{Q_k} \sum_{u\in(t_1,t_2]} \mathbf{1}_{\{Z_{t_1}^1=1\}} \mathbf{1}_{\{Z_u^1=1, Z_{u-}^1\neq 1\}} = 0. \tag{6.7}$$

Next consider the case when $Z_{t_1}^1 \neq 1$. The probability that the process Z^1 jumps to 1 before t_2 is equal to $1 - e^{-(t_2-t_1)/(k-1)}$, so

$$\mathbb{P}_{Q_k}\big(Z_{t_1}^1 \neq 1, S_1 \le t_2\big) = \frac{k-1}{k}\big(1 - e^{-(t_2-t_1)/(k-1)}\big), \tag{6.8}$$

and, consequently,

$$\lim_{k\to\infty} k\mathbb{E}_{Q_k} (\mathbf{1}_{\{Z_{t_1}^1\neq 1\}} \mathbf{1}_{\{S_1\le t_2\}}) = t_2 - t_1. \tag{6.9}$$

By the strong Markov property applied at S_1 and (6.7),

$$\lim_{k\to\infty} k\mathbb{E}_{Q_k} \sum_{m\geq 2} \mathbf{1}_{\{Z^1_{t_1}\neq 1\}} \mathbf{1}_{\{S_m\leq t_2\}} = 0. \tag{6.10}$$

We combine (6.7) and (6.9)–(6.10) to see that (6.5) holds, and therefore, (6.3) is true.

Given (6.3), in order to prove (6.4), it is necessary and sufficient to show that

$$\lim_{k\to\infty} \lim_{n\to\infty} \mathbb{E}_{Q_k} A(k,n)^2 \leq (t_2-t_1)^2. \tag{6.11}$$

Let $S^j_1, S^j_2, \dots$ denote the consecutive jumps of Z^j to the state 1 on the interval $(t_1, t_2]$. We have

$$\begin{aligned}
A(k,n)^2 &= \left(\frac{k}{n}\right)^2 \sum_{u,v\in[t_1,t_2]} \sum_{i,j=1}^n \mathbf{1}_{\{Z^i_u=1, Z^i_{u-}\neq 1, Z^j_v=1, Z^j_{v-}\neq 1\}} \\
&= \left(\frac{k}{n}\right)^2 \sum_{m,r\geq 1} \sum_{i,j=1}^n \mathbf{1}_{\{S^i_m\leq t_2, S^j_r\leq t_2\}} \\
&= \left(\frac{k}{n}\right)^2 \sum_{m,r\geq 1} \sum_{j=1}^n \mathbf{1}_{\{S^j_m\leq t_2, S^j_r\leq t_2\}} \\
&\quad + \left(\frac{k}{n}\right)^2 \sum_{m,r\geq 1} \sum_{i,j=1,i\neq j}^n \mathbf{1}_{\{S^i_m\leq t_2, S^j_r\leq t_2\}} \\
&\leq \left(\frac{k}{n}\right)^2 \sum_{j=1}^n \mathbf{1}_{\{S^j_1\leq t_2\}} + \left(\frac{k}{n}\right)^2 \sum_{m\geq 2} 2m \sum_{j=1}^n \mathbf{1}_{\{S^j_m\leq t_2\}} \\
&\quad + \left(\frac{k}{n}\right)^2 \sum_{m,r\geq 1} \sum_{i,j=1,i\neq j}^n \mathbf{1}_{\{S^i_m\leq t_2, S^j_r\leq t_2\}}.
\end{aligned} \tag{6.12}$$

Combining (6.6) and (6.8), we obtain

$$\mathbb{P}_{Q_k}\big(S^1_1\leq t_2\big) \leq (t_2-t_1)(k-1)^{-2} + \frac{k-1}{k}\big(1-e^{-(t_2-t_1)/(k-1)}\big),$$

which has a finite value for each k, so

$$\lim_{k\to\infty} \lim_{n\to\infty} \mathbb{E}_{Q_k}\left[\left(\frac{k}{n}\right)^2 \sum_{j=1}^n \mathbf{1}_{\{S^j_1\leq t_2\}}\right] = 0. \tag{6.13}$$

By (6.6) and the strong Markov property applied at S^j_1, we have for $m\geq 2$,

$$\mathbb{P}_{Q_k}\big(S^j_m\leq t_2\big) \leq (t_2-t_1)^{m-1}(k-1)^{-m},$$

so

$$\begin{aligned}
&\lim_{k\to\infty}\lim_{n\to\infty}\mathbb{E}_{Q_k}\left[\left(\frac{k}{n}\right)^2\sum_{m\geq 2}2m\sum_{j=1}^{n}\mathbf{1}_{\{S_m^j\leq t_2\}}\right] \\
&\qquad\leq \lim_{k\to\infty}\lim_{n\to\infty}\left[\left(\frac{k}{n}\right)^2\sum_{m\geq 2}2m\sum_{j=1}^{n}(t_2-t_1)^{m-1}(k-1)^{-m}\right]=0.
\end{aligned}\tag{6.14}$$

In view of (6.12)–(6.14), to complete the proof of (6.11), it remains to show that

$$\lim_{k\to\infty}\lim_{n\to\infty}\mathbb{E}_{Q_k}\left[\left(\frac{k}{n}\right)^2\sum_{m,r\geq 1}\sum_{i,j=1,i\neq j}^{n}\mathbf{1}_{\{S_m^i\leq t_2,S_r^j\leq t_2\}}\right]\leq(t_2-t_1)^2.\tag{6.15}$$

Since the joint distribution of (Z^i, Z^j) does not depend on i and j as long as $i\neq j$, (6.15) will follow once we prove

$$\lim_{k\to\infty}\mathbb{E}_{Q_k}\left[k^2\sum_{m,r\geq 1}\mathbf{1}_{\{S_m^1\leq t_2,S_r^2\leq t_2\}}\right]\leq(t_2-t_1)^2.\tag{6.16}$$

We will estimate the proportion of time that Z^1 and Z^2 spend in the same state. After the two processes meet, they spend an exponential amount of time together, with mean one, and then they jump at the same time. They jump to the same state with probability $1/(k-1)$ and if they do, they spend another period of exponential length in the same state. The sequence of jumps to the same state has geometric length with expectation $(k-1)/(k-2)$, so the total time the processes spend together before they separate has expectation $(k-1)/(k-2)$. When the processes travel through separate states, each one jumps to the state occupied by the other process at the rate $1/(k-1)$, so the waiting time for the next meeting at some state is exponential with mean $(k-1)/2$. It follows that in the long run, the proportion of time the two processes are in the same state is

$$\frac{(k-1)/(k-2)}{(k-1)/(k-2)+(k-1)/2}=\frac{2}{k}.\tag{6.17}$$

By symmetry, the proportion of time spent by the two processes in state 1 is $2/k^2$, so

$$\mathbb{P}_{Q_k}(Z_{t_1}^1=1,Z_{t_1}^2=1)=2/k^2.$$

This and the argument given in support of (6.6) can be combined to see that

$$\begin{aligned}
\mathbb{P}_{Q_k}(Z_{t_1}^1=Z_{t_1}^2=1,S_m^1\leq t_2)&\leq\frac{2}{k^2}\left(1-e^{-(t_2-t_1)/(k-1)}\right)^m \\
&\leq 2(t_2-t_1)^m(k-1)^{-m-2},
\end{aligned}$$

and, therefore,

$$
\begin{aligned}
\mathbb{E}_{Q_k}\Biggl[k^2 \sum_{m,r\ge 1} \mathbf{1}_{\{Z^1_{t_1}=Z^2_{t_1}=1, S^1_m\le t_2, S^2_r\le t_2\}}\Biggr] &\le 2\mathbb{E}_{Q_k}\Biggl[k^2 \sum_{1\le r\le m} \mathbf{1}_{\{Z^1_{t_1}=Z^2_{t_1}=1, S^1_m\le t_2, S^2_r\le t_2\}}\Biggr] \\
&\le 2\mathbb{E}_{Q_k}\Biggl[mk^2 \sum_{m\ge 1} \mathbf{1}_{\{Z^1_{t_1}=Z^2_{t_1}=1, S^1_m\le t_2\}}\Biggr] \\
&\le 4k^2 \sum_{m\ge 1} m(t_2-t_1)^m (k-1)^{-m-2}.
\end{aligned}
$$

This implies that

$$
\lim_{k\to\infty} \mathbb{E}_{Q_k}\Biggl[k^2 \sum_{m,r\ge 1} \mathbf{1}_{\{Z^1_{t_1}=Z^2_{t_1}=1, S^1_m\le t_2, S^2_r\le t_2\}}\Biggr] = 0. \tag{6.18}
$$

We will now estimate $\mathbb{P}_{Q_k}(Z^1_{t_1}=1, S^1_m \le t_2, S^2_r \le t_2)$. By symmetry, $\mathbb{P}_{Q_k}(Z^1_{t_1} = 1) = 1/k$. Consider the case $m = r = 1$, and suppose that $Z^1_{t_1} = 1$. After the process Z^1 jumps to some other state, it has probability less than $1 - e^{-(t_2-t_1)/(k-1)}$ of jumping to 1 in the remaining time in the interval $[t_1, t_2]$. The probability that Z^2 will jump to 1 from some other state during $[t_1, t_2]$ is bounded by $1 - e^{-(t_2-t_1)/(k-1)}$, no matter where Z^2 is at the time t_1. Hence, the probability that at least one of the processes Z^1 or Z^2 jumps to 1 from some other state during $[t_1, t_2]$ is bounded by $2(1 - e^{-(t_2-t_1)/(k-1)})$. Now we consider two cases. The first one is that at the time of the first jump of Z^1 or Z^2 to 1 from some other state; the other process jumps as well. The conditional probability that the second one will also jump to 1 is $1/(k-1)$. The second case is that the other process does not jump at the same time. The probability that it will jump to 1 in the remaining time in $[t_1, t_2]$ is bounded by $1 - e^{-(t_2-t_1)/(k-1)}$. Altogether,

$$
\begin{aligned}
&\mathbb{P}_{Q_k}\bigl(Z^1_{t_1}=1, S^1_1 \le t_2, S^2_1 \le t_2\bigr) \\
&\quad \le \frac{1}{k}\cdot 2\bigl(1-e^{-(t_2-t_1)/(k-1)}\bigr)\biggl(\frac{1}{k-1} + 1 - e^{-(t_2-t_1)/(k-1)}\biggr) \\
&\quad \le c(t_1,t_2)\frac{1}{(k-1)^3}.
\end{aligned} \tag{6.19}
$$

The same argument that proves (6.6) gives for any n and $m \ge 1$,

$$
\begin{aligned}
\mathbb{P}_{Q_k}\bigl(Z^1_{t_1}=1, S^1_m \le t_2\bigr) &\le \frac{1}{k}\bigl(1-e^{-(t_2-t_1)/(k-1)}\bigr)^m \\
&\le (t_2-t_1)^m (k-1)^{-m-1}
\end{aligned} \tag{6.20}
$$

and

$$
\begin{aligned}
\mathbb{P}_{Q_k}\bigl(Z^1_{t_1}=1, S^2_m \le t_2\bigr) &\le \frac{1}{k}\bigl(1-e^{-(t_2-t_1)/(k-1)}\bigr)^m \\
&\le (t_2-t_1)^m (k-1)^{-m-1}.
\end{aligned} \tag{6.21}
$$

We combine (6.19)–(6.21) to see that

$$\begin{aligned}
&\mathbb{E}_{Q_k}\left[k^2 \sum_{m,r\ge 1} \mathbf{1}_{\{Z^1_{t_1}=1, S^1_m\le t_2, S^2_r\le t_2\}}\right] \\
&\quad\le \mathbb{E}_{Q_k}\big[k^2 \mathbf{1}_{\{Z^1_{t_1}=1, S^1_1\le t_2, S^2_1\le t_2\}}\big] + \mathbb{E}_{Q_k}\left[k^2 \sum_{1\le r\le m, m\ge 2} \mathbf{1}_{\{Z^1_{t_1}=1, S^1_m\le t_2, S^2_r\le t_2\}}\right] \\
&\qquad + \mathbb{E}_{Q_k}\left[k^2 \sum_{1\le m\le r, r\ge 2} \mathbf{1}_{\{Z^1_{t_1}=1, S^1_m\le t_2, S^2_r\le t_2\}}\right] \\
&\quad\le \mathbb{E}_{Q_k}\big[k^2 \mathbf{1}_{\{Z^1_{t_1}=1, S^1_1\le t_2, S^2_1\le t_2\}}\big] + \mathbb{E}_{Q_k}\left[mk^2 \sum_{m\ge 2} \mathbf{1}_{\{Z^1_{t_1}=1, S^1_m\le t_2\}}\right] \\
&\qquad + \mathbb{E}_{Q_k}\left[rk^2 \sum_{r\ge 2} \mathbf{1}_{\{Z^1_{t_1}=1, S^2_r\le t_2\}}\right] \\
&\quad\le k^2 c(t_1,t_2)\frac{1}{(k-1)^3} + 2k^2 \sum_{m\ge 2} m(t_2-t_1)^m (k-1)^{-m-1}.
\end{aligned}$$

This implies that

$$\lim_{k\to\infty} \mathbb{E}_{Q_k}\left[k^2 \sum_{m,r\ge 1} \mathbf{1}_{\{Z^1_{t_1}=1, S^1_m\le t_2, S^2_r\le t_2\}}\right] = 0. \tag{6.22}$$

By symmetry,

$$\lim_{k\to\infty} \mathbb{E}_{Q_k}\left[k^2 \sum_{m,r\ge 1} \mathbf{1}_{\{Z^2_{t_1}=1, S^1_m\le t_2, S^2_r\le t_2\}}\right] = 0. \tag{6.23}$$

It follows from (6.17) that $\mathbb{P}_{Q_k}(Z^1_{t_1} = Z^2_{t_1} \ne 1) = 2(k-1)/k^2$. The reasoning completely analogous to that given in the case when $Z^1_{t_1} = 1$ yields

$$\lim_{k\to\infty} \mathbb{E}_{Q_k}\left[k^2 \sum_{m,r\ge 1} \mathbf{1}_{\{Z^1_{t_1}=Z^2_{t_1}\ne 1, S^1_m\le t_2, S^2_r\le t_2\}}\right] = 0. \tag{6.24}$$

Finally, consider the event $F := \{Z^1_{t_1} \ne Z^2_{t_1}, Z^1_{t_1} \ne 1, Z^2_{t_1} \ne 1\}$. The probability that Z^1 will jump to 1 during $[t_1, t_2]$ is equal to $1 - e^{-(t_2-t_1)/(k-1)}$. Let $\tau = \inf\{t \ge t_1 : Z^1_t = Z^2_t\}$ ($\tau = t_2$ if the two processes do not meet before t_2). We have

$$\begin{aligned}
&\mathbb{P}_{Q_k}(F, S^1_1 \le t_2, S^2_1 \le t_2) \\
&\quad = \mathbb{P}_{Q_k}(F, S^1_1 < \tau < S^2_1 \le t_2) + \mathbb{P}_{Q_k}(F, S^2_1 < \tau < S^1_1 \le t_2) \\
&\qquad + \mathbb{P}_{Q_k}(F, \tau < S^1_1 \le t_2, \tau < S^2_1 \le t_2) \\
&\qquad + \mathbb{P}_{Q_k}(F, S^1_1 \le t_2, S^2_1 \le \tau \wedge t_2).
\end{aligned} \tag{6.25}$$

Our usual estimates give

$$\mathbb{P}_{Q_k}\big(F, S_1^2 < \tau < S_1^1 \le t_2\big) \le \big(1 - e^{-(t_2-t_1)/(k-1)}\big)^3 \le (t_2 - t_1)^3 (k-1)^{-3},$$

so

$$\lim_{k\to\infty} \mathbb{E}_{Q_k}\big[k^2 \mathbf{1}_{F\cap\{S_1^2<\tau<S_1^1\le t_2\}}\big] \le \lim_{k\to\infty} k^2 (t_2-t_1)^3 (k-1)^{-3} = 0, \tag{6.26}$$

and, by symmetry,

$$\lim_{k\to\infty} \mathbb{E}_{Q_k}\big[k^2 \mathbf{1}_{F\cap\{S_1^1<\tau<S_1^2\le t_2\}}\big] = 0. \tag{6.27}$$

Given $\{Z_t^1, t \in [t_1, t_2]\}$, the conditional probability that Z^2 jumps to 1 before or at time $\tau \wedge t_2$ is bounded by $1 - e^{-(t_2-t_1)/(k-1)}$. It follows that

$$\mathbb{P}_{Q_k}\big(F, S_1^1 \le t_2, S_1^2 \le \tau \wedge t_2\big) \le \big(1 - e^{-(t_2-t_1)/(k-1)}\big)^2 \le (t_2 - t_1)^2 (k-1)^{-2}$$

and

$$\begin{aligned} \lim_{k\to\infty} \mathbb{E}_{Q_k}\big[k^2 \mathbf{1}_{F\cap\{S_1^1\le t_2, S_1^2\le \tau\wedge t_2\}}\big] &\le \lim_{k\to\infty} k^2 (t_2-t_1)^2 (k-1)^{-2} \\ &= (t_2-t_1)^2. \end{aligned} \tag{6.28}$$

The probability that, given F, the coupling time τ will occur before t_2 is bounded by $1 - e^{-(t_2-t_1)/(k-1)} \le (t_2 - t_1)/(k-1)$, so using the strong Markov property at τ, the case of $F \cap \{\tau < S_1^1 \le t_2, \tau < S_1^2 \le t_2\}$ is reduced to that in (6.24), and we obtain the following bound:

$$\lim_{k\to\infty} \mathbb{E}_{Q_k}\big[k^2 \mathbf{1}_{F\cap\{\tau<S_1^1\le t_2, \tau<S_1^2\le t_2\}}\big] = 0. \tag{6.29}$$

In view of (6.25), estimates (6.26)–(6.29) yield

$$\lim_{k\to\infty} \mathbb{E}_{Q_k}\big[k^2 \mathbf{1}_{F\cap\{S_1^1\le t_2, S_1^2\le t_2\}}\big] \le (t_2-t_1)^2. \tag{6.30}$$

A similar analysis gives

$$\lim_{k\to\infty} \mathbb{E}_{Q_k}\big[k^2 \mathbf{1}_{F\cap\{S_2^1\le t_2, S_1^2\le t_2\}}\big] = 0 \tag{6.31}$$

and

$$\lim_{k\to\infty} \mathbb{E}_{Q_k}\big[k^2 \mathbf{1}_{F\cap\{S_1^1\le t_2, S_2^2\le t_2\}}\big] = 0. \tag{6.32}$$

Our usual arguments give for $m \ge 0$,

$$\mathbb{P}_{Q_k}\big(S_m^1 \le t_2 | F\big) \le \big(1 - e^{-(t_2-t_1)/(k-1)}\big)^m \le 2(t_2 - t_1)^m (k-1)^{-m},$$

so

$$\begin{aligned} \mathbb{E}_{Q_k}\Bigg[k^2 \sum_{m,r\ge 3} \mathbf{1}_{F\cap\{S_m^1\le t_2, S_r^2\le t_2\}}\Bigg] &\le \mathbb{E}_{Q_k}\Bigg[mk^2 \sum_{m\ge 3} \mathbf{1}_{F\cap\{S_m^1\le t_2\}}\Bigg] \\ &\le mk^2 \sum_{m\ge 3} 2(t_2-t_1)^m (k-1)^{-m}, \end{aligned}$$

and, therefore,

$$\lim_{k\to\infty} \mathbb{E}_{Q_k}\left[k^2 \sum_{m,r\geq 3} \mathbf{1}_{F\cap\{S_m^1\leq t_2, S_r^2\leq t_2\}}\right] = 0.$$

This and (6.30)–(6.32) give

$$\lim_{k\to\infty} \mathbb{E}_{Q_k}\left[k^2 \sum_{m,r\geq 1} \mathbf{1}_{F\cap\{S_m^1\leq t_2, S_r^2\leq t_2\}}\right] \leq (t_2 - t_1)^2.$$

We deduce (6.16) from the last estimate, (6.18) and (6.22)–(6.24). This completes the proof. □

PROOF OF COROLLARY 6.3. (i) The process $\{N_t^1, t \geq 0\}$ is Poisson with rate one. It is routine to check that the exponential distribution with mean 1 is the stationary distribution for the process $t \to t - T_t^1$. This easily implies part (i) of the corollary.

(ii) Processes N^j, $j = 1, \dots, n$, are independent, so processes $\{t - T_t^j, t \geq 0\}$, $j = 1, \dots, n$, are independent. This and Theorem 6.2 imply part (ii). □

7. Earthworm effect. An "earthworm" model was introduced in [7]. The model involves a ball moving in a Euclidean torus which pushes "soil particles" aside. The motion of the center of the ball is that of Brownian motion. The paper [7] contains a result which suggests that in dimensions 3 and higher, the "spherical earthworm" does not compactify the soil on a global scale, assuming that the torus diameter is much larger than that of the ball (the result is asymptotic, in other words). The result in [7] does not answer a number of conjectures stated in that paper. Finishing that research program appears to involve major technical challenges. In this article, we will present a discrete version of the earthworm model and a result that is closer to the conjectures stated in [7], at least at the heuristic level. We will show that if $G = C_n^d$ is a torus with a large diameter, then in the long run, the soil will be uniformly distributed over G, in an appropriate sense, as a result of earthworm's stirring action.

We now present the rigorous version of the "earthworm" model. Given a graph G with a vertex set V, we will define the mass process $\mathcal{M}_t = (M_t^{v_1}, M_t^{v_2}, \dots)$, with an evolution different than that in the previous sections of the paper. Suppose that B_t (the "earthworm") is a simple random walk on G, that is, B_t is a Markov process which takes values in V, stays constant for an exponential (mean 1) amount of time, and jumps to a uniformly chosen nearest neighbor at the end of the exponential holding time. At the time t of a jump of B, M^{B_t} jumps to 0. At the same time, the mass $M_{t-}^{B_t}$ is "distributed" to all adjacent sites, that is, for every vertex x connected to B_t by an edge, the process M^x increases by $M_{t-}^{B_t}/d_v$, that is, $M_t^x = M_{t-}^x + M_{t-}^{B_t}/d_v$. The processes M^v are constant between the jumps of B. The mass M^v can jump only when B jumps to v or a neighbor of v in the graph G.

Let $\mathbf{M}$ be the empirical measure for the process $\{M_t^v\}_{v\in V}$, that is, $\mathbf{M}_t = \sum_{v\in V} \delta_{M_t^v}$, where δ_x stands for the measure with a unit atom at x ("Dirac's delta").

Note that in the following theorem, by the symmetry of the torus, the initial position of B is irrelevant, so we may assume that $B_0 = \mathbf{0} := (0, \dots, 0)$.

THEOREM 7.1. *Fix $d \ge 1$, and let $\mathbf{M}_t^n$ be the empirical measure process for the earthworm process on the graph $G = C_n^d$. Assume that $M_0^v = 1/n^d$ for $v \in V$ (hence $\sum_{v\in V} M_0^v = 1$).*

(i) *For every n, the random measures $\mathbf{M}_t^n$ converge weakly to a random measure $\mathbf{M}_\infty^n$, when $t \to \infty$.*

(ii) *For $R \subset \mathbb{R}^d$ and $a \in \mathbb{R}$, let $aR = \{x \in \mathbb{R}^d : x = ay \text{ for some } y \in R\}$ and $\widehat{\mathbf{M}}_\infty^n(R) = \mathbf{M}_\infty^n(nR)$. When $n \to \infty$, the random measures $\widehat{\mathbf{M}}_\infty^n$ converge weakly to the random measure equal to, a.s., the uniform probability measure on $[0, 1]^d$.*

PROOF. (i) The proof of Theorem 3.2 applies in the present case, with some minor modification accounting for the fact that the mass redistribution mechanism is given by B rather than Poisson processes N^x. Hence, there exists a unique stationary distribution Q for $(\mathcal{M}_t, B_t)$. Under Q, $\mathbf{M}_t^n$ has distribution $\mathbf{M}_\infty^n$.

(ii) Let $|R|$ denote the d-dimensional Lebesgue measure of $R \subset \mathbb{R}^d$. To prove part (ii) of the theorem, it will suffice to show that for every fixed rectangle $R \subset [0, 1]^d$ with rational vertices, $\lim_{n\to\infty} \widehat{\mathbf{M}}_\infty^n(R) = |R|$, in probability. It will be enough to show that

$$\lim_{n\to\infty} \mathbb{E}_Q \widehat{\mathbf{M}}_\infty^n(R) = |R| \tag{7.1}$$

and

$$\lim_{n\to\infty} \operatorname{Var}_Q \widehat{\mathbf{M}}_\infty^n(R) = 0. \tag{7.2}$$

By symmetry, $\mathbb{E}_Q M_0^x = \mathbb{E}_Q M_0^y$ for all $x, y \in V$. Since $\sum_{x\in V} M_0^x = 1$ under Q, we must have $\mathbb{E}_Q M_0^x = 1/n^d$. By abuse of notation, we give $|\cdot|$ another meaning—it will denote the cardinality of an (at most) countable set. We have

$$\lim_{n\to\infty} \mathbb{E}_Q \widehat{\mathbf{M}}_\infty^n(R) = \lim_{n\to\infty} \mathbb{E}_Q \mathbf{M}_\infty^n(nR) = \lim_{n\to\infty} \frac{1}{n^d} |nR \cap V| = |R|,$$

and thus (7.1) is proved.

Let Z and $\widetilde{Z}$ be defined as Z^1 and Z^2 in Definition 2.1. In particular, $\mathbb{P}(Z_0 = x) = \mathbb{P}(\widetilde{Z}_0 = x) = M_0^x$ for $x \in V$. However, note that in the present case, the process $\{Z_t, t \ge 0\}$ jumps at a time t if and only if B jumps at the time t and $B_t = Z_{t-}$. A similar remark applies to $\{\widetilde{Z}_t, t \ge 0\}$. Note that the jump times of Z and $\widetilde{Z}$ are defined by the same process B.

The state space for the process $(Z, \widetilde{Z})$ is finite, so it has a stationary distribution. The stationary distribution is unique because all states communicate. We will next estimate the stationary probabilities, in the asymptotic sense, when $n \to \infty$.

Let $\overline{Z}_t = Z_t - \widetilde{Z}_t \in V$ (in the sense of group operations on the Cayley graph). Although $\overline{Z}_t$ is not a Markov process (as far as we can tell), it is clear how to define a discrete time Markov chain $\{U_j, j \geq 1\}$ embedded in $\overline{Z}_t$.

For $x \in V$, let $\mathcal{B}(x, r)$ denote the closed ball in V with center x and radius r, relative to the graph distance.

Let $\mathbf{h}$ be set of all vertices $(f_1, \dots, f_d)$ such that $|f_i| = 1$ for some i, and $f_m = 0$ for all $m \neq i$. It has been shown in the proof of Theorem 5.1 that the stationary distribution $\{\pi_x, x \in V\}$ for U has the following form. For some normalizing constant $c_1 > 0$, $\pi_x = (1 - \frac{1}{2d})c_1$ for all $x \in \mathbf{h}$ and $\pi_x = c_1$ for all other $x \in V$.

Although $\overline{Z}$ is not a Markov process, $(\overline{Z}, Z, \widetilde{Z})$ is. We will consider the process $(\overline{Z}, Z, \widetilde{Z})$ in the stationary regime. Let $\overline{Z}$ have the corresponding marginal distribution. We will estimate the proportion of time that $\overline{Z}$ spends in different states. For each state x, we will estimate the product of π_x and the expected amount of time between the time τ_1 of the first jump of $\overline{Z}$ to x and the time τ_2 of the next jump. Let us call the random time between these jumps $\tau_x = \tau_2 - \tau_1$. Hence we will estimate $\mathbb{E}\tau_x$.

Consider $x \in \mathcal{B}(\mathbf{0}, 2)^c$ and any two neighbors y and z of x. We have $\pi_y = \pi_z$, so the probability that the process $\overline{Z}$ jumps to x from y is equal to the probability that the process $\overline{Z}$ jumps to x from z. Hence, $\mathbb{P}(B_{\tau_1} = x) = \mathbb{P}(B_{\tau_1} = y)$ for any neighbors x and y of Z_{τ_1} and $\widetilde{Z}_{\tau_1}$. The time τ_x is the same as the waiting time for the first hit of $\{Z_{\tau_1}, \widetilde{Z}_{\tau_1}\}$ after time τ_1, for B.

Let K be the set of all neighbors of Z_{τ_1} and $\widetilde{Z}_{\tau_1}$. We have shown that the distribution of B_{τ_1} is uniform on K. It follows from [1], Corollary 24, page 21, Chapter 2, that the expected time until B hits $\{Z_{\tau_1}, \widetilde{Z}_{\tau_1}\}$ is $|V|/2 - 1$. This implies that $\mathbb{E}\tau_x = |V|/2 - 1$. Thus for any $x, y \in \mathcal{B}(\mathbf{0}, n_1)^c$, we have $\mathbb{E}\tau_x = \mathbb{E}\tau_y$. This and the fact that $\pi_x = \pi_y$ imply that, under the stationary distribution, for $x, y \in \mathcal{B}(\mathbf{0}, 2)^c$, $\mathbb{P}(\overline{Z}_0 = x) = \mathbb{P}(\overline{Z}_0 = y)$. Therefore, if $x - y \in \mathcal{B}(\mathbf{0}, 2)^c$,

$$\mathbb{P}_Q(Z_0 = x, \widetilde{Z}_0 = y) = \mathbb{P}_Q(Z_0 = x)\mathbb{P}_Q(\widetilde{Z}_0 = y). \tag{7.3}$$

For $x \in \mathcal{B}(\mathbf{0}, 2)$ we have a rough bound $\mathbb{E}\tau_x \leq c_1 n^d$, which yields for $x - y \in \mathcal{B}(\mathbf{0}, 2)$,

$$\mathbb{P}_Q(Z_0 = x, \widetilde{Z}_0 = y) \leq c_2 \mathbb{P}_Q(Z_0 = x)\mathbb{P}_Q(\widetilde{Z}_0 = y). \tag{7.4}$$

Let $\mathcal{G}_t = \sigma(\mathcal{M}_s, 0 \leq s \leq t) = \sigma(\mathcal{M}_0, B_s, 0 \leq s \leq t)$. We have for $x \in V$,

$$\mathbb{P}_Q(Z_0 = x | \mathcal{G}_0) = M_0^x.$$

The processes Z_t and $\widetilde{Z}_t$ are conditionally independent given $\mathcal{G}_t$, so for $x, y \in V$,

$$\mathbb{P}_Q(Z_t = x, \widetilde{Z}_t = y | \mathcal{G}_t) = M_t^x M_t^y.$$

By stationarity, for $x, y \in V$,

$$\mathbb{P}_Q(Z_0 = x, \widetilde{Z}_0 = y | \mathcal{G}_0) = M_0^x M_0^y.$$

Thus

$$\mathbb{E}_Q\big(M_0^x M_0^y\big) = \mathbb{E}_Q \mathbb{P}_Q(Z_0 = x, \widetilde{Z}_0 = y | \mathcal{G}_0) = \mathbb{P}_Q(Z_0 = x, \widetilde{Z}_0 = y).$$

We obtain

$$\begin{aligned}\mathbb{E}_Q\big(\mathbf{M}_\infty^n(nR)\big)^2 &= \sum_{x,y \in nR} \mathbb{E}_Q\big(M_0^x M_0^y\big) \\ &= \sum_{\substack{x,y \in nR \\ x-y \in \mathcal{B}(\mathbf{0},2)^c}} \mathbb{E}_Q\big(M_0^x M_0^y\big) + \sum_{\substack{x,y \in nR \\ x-y \in \mathcal{B}(\mathbf{0},2)}} \mathbb{E}_Q\big(M_0^x M_0^y\big).\end{aligned} \tag{7.5}$$

It follows from (7.4) that

$$\begin{aligned}\sum_{\substack{x,y \in nR \\ x-y \in \mathcal{B}(\mathbf{0},2)}} \mathbb{E}_Q\big(M_0^x M_0^y\big) &= \sum_{\substack{x,y \in nR \\ x-y \in \mathcal{B}(\mathbf{0},2)}} \mathbb{P}_Q(Z_0 = x, \widetilde{Z}_0 = y) \\ &\le \sum_{\substack{x,y \in nR \\ x-y \in \mathcal{B}(\mathbf{0},2)}} c_2 \mathbb{P}_Q(Z_0 = x) \mathbb{P}_Q(\widetilde{Z}_0 = y) \\ &\le c_3 |R| / n^d.\end{aligned} \tag{7.6}$$

We use (7.3) to see that

$$\begin{aligned}\sum_{\substack{x,y \in nR \\ x-y \in \mathcal{B}(\mathbf{0},2)^c}} \mathbb{E}_Q\big(M_0^x M_0^y\big) &= \sum_{\substack{x,y \in nR \\ x-y \in \mathcal{B}(\mathbf{0},2)^c}} \mathbb{P}_Q(Z_0 = x, \widetilde{Z}_0 = y) \\ &\le \sum_{\substack{x,y \in nR \\ x-y \in \mathcal{B}(\mathbf{0},2)^c}} \mathbb{P}_Q(Z_0 = x) \mathbb{P}_Q(\widetilde{Z}_0 = y) \le |R|^2.\end{aligned}$$

This, (7.5) and (7.6) give

$$\lim_{n\to\infty} \mathbb{E}_Q\big(\mathbf{M}_\infty^n(nR)\big)^2 \le \lim_{n\to\infty} \big(c_3|R|/n^d + |R|^2\big) = |R|^2.$$

We obtain, $\lim_{n\to\infty} \mathbb{E}_Q(\mathbf{M}_\infty^n(nR))^2 \le |R|^2$. This shows (7.2), thus completing the proof. $\square$

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