1 Abstract

These purpose of these notes is to provide a gentle introduction to the theory of amenable groups. We begin by motivating amenable groups where much of our exposition is from Paterson’s text *Amenability*. We then consider a few equivalent characterizations of amenability where the proofs of the equivalences come from Brown and Ozawa’s text *C*-Algebras and Finite-Dimensional Approximations. We then apply our theory to group *C*-algebras where we show amenability holds if and only if the full *C*-algebra agrees with the reduced *C*-algebra where we follow Davidson’s text *C*-Algebras by Example.

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2 Amenability

In this section we introduce the definition of amenability. We then prove a few equivalent characterizations of amenability.

2.1 Amenable Groups

Many introductory measure theory textbooks begin by positing the existence of a measure defined on all subsets of $\mathbb{R}^n$ that posses translation invariance, assigns measure 1 to the unit
Definition 1. A mean on a group $G$ is a finitely additive positive probability measure on $G$. A group $G$ is (left) amenable if we can equip it with a left invariant mean.

In the 1940’s, M. M. Day observed that a mean $\mu$ can be regarded as an element $m$ of $L^\infty(G)^*$ via

$$m\left(\sum_{i=1}^{n} \alpha_i \chi_{E_i}\right) := \sum_{i=1}^{n} \alpha_i \mu(E_i)$$

for simple function and then extending to all of $L^\infty(G)$ by considering point-wise limits of simple functions. This observation is key because it allows the ability to study amenable groups with the machinery of functional analysis. Denote the set of means by

$$M(G) := \{m \in L^\infty(G)^*|m(1) = \|m\|\}$$

and the left invariant means by $L(G)$. With all the benefits of viewing a mean as a functional on $L^\infty(G)$, we have the drawback that now we have to try to understand $L^\infty(G)^*$ for a locally compact group $G$. Hence, it is advantageous to get a hold of a “nice” subset of $L^\infty(G)^*$. To do this we will need to recall some functional analysis.

Recall that all Banach spaces embed into their double dual by evaluation. So that $L^1(G) \hookrightarrow L^1(G)^{**}$ by $f \mapsto \hat{f}$ where $\hat{f}(f') = \int f f' d\lambda$. On the other hand, by Riesz Representation we know that $L^\infty(G)$ can be identified with $L^1(G)^*$ by integration i.e. $\phi \mapsto I_\phi$ where $I_\phi(f) = \int f \phi d\lambda$. Hence every element $f \in L^1(G)$ corresponds to an element in $L^\infty(G)^*$ by $\int f \phi d\lambda$ for $\phi \in L^\infty(G)$. Let us consider those elements in $L^1(G)$ that land in $M(G)$. Denote this set by $P(G) = \{f \in L^1(G)|\hat{f} \in M(G)\}$. This is the “nice” set we want and the following proposition explains why.

Proposition 2. (i) A functional $m \in L^\infty(G)^*$ is a mean if and only if $m(1) = 1$ and $m(\phi) \geq 0$ for all $\phi \geq 0$ in $L^\infty(G)$.

(ii) The set $P(G) = \{f \in L^1(G)|f \geq 0$ and $\int f d\lambda = 1\}$ and $\hat{P(G)}$ is weak* dense in $M(G)$.

Remark 3. We have established a correspondence between finitely additive probability measures and means on $L^\infty(G)^*$. How does translation invariance transfer from the means? For $\phi \in L^\infty(G)$ and $x \in G$ define $\phi x$ by $\phi x(y) = \phi(xy)$. Then translation invariance of $\mu$ in terms of the linear functional $m$ becomes the relation

$$m(\phi x) = m(\phi) \text{ for } \phi \in L^\infty(G) \text{ and } x \in G.$$
We consider some examples and non-examples of amenable groups.

**Example 1.** Let $G$ be any compact group. Then Haar measure is a left invariant mean on $G$ so that all compact groups are amenable. In particular, every finite group is amenable.

**Example 2.** Let $G = \mathbb{Z}$. By our previous proposition $P(G)$ has its typical elements of the form $\sum_{l=-\infty}^{\infty} c_l \delta_l$ where $c_l \geq 0$ and $\sum c_l = 1$. To show $\mathbb{Z}$ is amenable we will aim to find a sequence in $P(\mathbb{Z})$ with at least one weak* limit point. By the previous proposition we will know that this limit point will be a mean and we will check that it is left invariant. Consider

$$f_n = \frac{1}{2n+1} \sum_{l=-n}^{n} \delta_l,$$

Then for $\phi \in L^\infty(\mathbb{Z})$ and $s \geq 0$ we have

$$|\hat{f}_n(\phi s) - \hat{f}_n(\phi)| = |I_{\phi s}(f_n) - I_\phi(f_n)| = \left| \frac{1}{2n+1} \sum_{l=-n}^{n} (I_{\phi s}(\delta_l) - I_\phi(\delta_l)) \right|$$

$$= \frac{1}{2n+1} \left| \sum_{l=-n}^{n} \left( \sum_{r \in \mathbb{Z}} (\phi s \delta_l(r) - \phi \delta_l(r)) \right) \right| = \frac{1}{2n+1} \left| \sum_{l=-n}^{n} \phi(s+l) - \phi(l) \right|$$

$$= \frac{1}{2n+1} \left| \sum_{l=n+1}^{n+s} \phi(l) - \sum_{l=-n}^{1} \phi(1) \right| \leq \frac{1}{2n+1} \|\phi\| 2(s-1) \to 0$$

A similar calculation holds for $s < 0$. Hence, since every limit point of $(\hat{f}_n)$ is a mean and the above calculation shows that it will be invariant, we conclude that $\mathbb{Z}$ is amenable.

**Example 3.** Let $G = \mathbb{R}$. It is natural to expect that a similar argument should hold for $\mathbb{R}$ but that we should have a “continuous” version of the $\mathbb{Z}$ case. We claim $f_n = \frac{1}{2n} \chi_{[-n,n]}$ will have every limit point of $(\hat{f}_n)$ a invariant mean. We leave the details to the reader.

**Example 4.** Let $G = \mathbb{F}_2$ be the free group on two generators, say $a, b$. We claim that $\mathbb{F}_2$ is not amenable. Indeed, suppose there existed a left invariant mean $m$. Let $W_x$ denote the set of reduced words beginning with $x$. Then, since $m$ is a mean,

$$1 = m(\mathbb{F}_2) = m(W_a) + m(W_{a^{-1}}) + m(W_b) + m(W_{b^{-1}}) + m(\{e\}).$$

On the other hand, by invariance of $m$,

$$1 = m(W_a) + m(aW_{a^{-1}}) = m(W_a) + m(W_{a^{-1}}).$$

Similarly, $1 = m(W_b) + m(W_{b^{-1}})$. Plugging in these identities into the first equation, we obtain

$$1 = m(\mathbb{F}_2) = 2 + m(\{e\})$$

which is impossible since $m$ is a positive measure.
2.2 Approximate Left Invariant Means

So far we have exploited the density of $\hat{P}(G)$ in $M(G)$ to get left invariant means. It would be nice to have a characterization of amenability that stays completely in $P(G)$. To do this we consider the action $G$ on $L^1(G)$ given by translation. That is, $x * f(y) = f(x^{-1}y)$ for $x \in G$ and $f \in L^1(G)$.

Theorem 4. A group $G$ is amenable if and only if there is an approximate left invariant mean. That is, there is a sequence $(f_n) \subseteq P(G)$ such that $\|x * f_n - f_n\|_1 \to 0$ for every $x \in G$.

Proof. The backwards direction is just a matter of unraveling definitions. Consider any weak* limit of $(f_n)$. Let $x \in G$ and $\phi \in L^\infty(G)$. Then,

$$|\hat{f}_n(x) - \hat{f}(x)| = |I_{xy}(f_n) - I_{\phi}(f_n)| = \left| \int \phi(xy)f_n(y)d\lambda(y) - \int \phi(y)f_n(y)d\lambda(y) \right|$$

$$= \left| \int \phi(y)(f_n(x^{-1}y) - f_n(y))d\lambda(y) \right| \leq ||\phi||_\infty \int |f_n(x^{-1}y) - f_n(y)|d\lambda(y)$$

$$= ||\phi||_\infty \|x * f_n - f_n\|_1 \to 0.$$

The forward direction is a matter of testing whether you remember your basic functional analysis. Suppose $G$ is amenable with mean $m$. Since $\hat{P}(G)$ is weak* dense in $M(G)$, then we can choose $(f_n) \subseteq P(G)$ so that $\hat{f}_n \to m$ in the weak* topology. Notice that for every $x \in G$, $x * f_n - f_n \to 0$ weakly in $L^1(G)$. Indeed, for $\phi \in L^\infty(G)$, we have by the invariance of $m$,

$$\left| \int \phi(y)(f_n(x^{-1}y) - f_n(y))dy \right| \leq \left| \int f_n(y) - m(\phi x) \right| + \left| m(\phi) - \int \phi(y)f_n(y)dy \right| \to 0.$$

We conclude that the weak closure of $\{x * f - f | f \in P(G)\}$ contains 0. But this set is clearly convex so that the weak closure and norm closure coincide. Hence, there is an approximate left invariant mean. \qed

2.3 Folner Conditions

For $K \subseteq G$ compact and $f = \chi_K/\lambda(K)$, let us consider what the approximate left invariant mean condition becomes:

$$\|x * f - f\|_1 = \frac{1}{\lambda(K)} \int |\chi_K - \chi^x_K|d\lambda = \frac{\lambda(xK \Delta K)}{\lambda(K)}.$$

That such an expression should come up so naturally motivates the following definitions. For simplicity, we will assume that $G$ is countable and discrete for the reminder of this section.

Definition 5. A sequence $F_n \subseteq G$ of finite sets such that

$$\frac{|xF_n \Delta F_n|}{|F_n|} \to 0$$

for every $x \in G$ is called a Folner sequence.

Definition 6. We say $G$ satisfies a Folner condition if for every finite set $S$ and $\epsilon > 0$ there is a finite set $F \subseteq G$ with

$$\max_{x \in S} \frac{|xF \Delta F|}{|F|} < \epsilon.$$
It is not hard to see that these notions are equivalent.

**Proposition 7.** A group $G$ has a Folner sequence if and only if it satisfies a Folner condition.

**Proof.** The forward direction is trivial. For the backwards direction write $G = \bigcup_n S_n$ where each $S_n$ is finite and they satisfy $S_1 \subseteq S_2 \subseteq \cdots$. Then take $\epsilon = 1/n$ and $S = S_n$. By the Folner condition, we obtain some $F_n$ with

$$\frac{|xF_n \triangle F_n|}{|F_n|} < \frac{1}{n}$$

for all $s \in S_n$. Then for any $x \in G$, there is some $S_n$ for $n$ large enough that contains $x$ and satisfies

$$\frac{|xF_n \triangle F_n|}{|F_n|} < \frac{1}{n} \to 0 \text{ as } n \to \infty.$$

□

We have already seen examples of Folner sequences e.g. $[-n,n] \cap \mathbb{Z}$ for $\mathbb{Z}$ and $[-n,n]$ for $\mathbb{R}$ hinting at some connection between amenability and of Folner sequences. In fact, the connection between the existence of Folner sequences and of approximate left invariant means was already made at the beginning of the section by taking $f_n = \chi_{F_n}/|F_n|$ where $F_n$ is a Folner sequence. In other words, the existence of a Folner sequences (or of a Folner condition) is obviously stronger than that of an approximate left invariant mean (and consequently stronger than amenability.) It turns out that these notions are equivalent.

**Theorem 8.** If $G$ has an approximate left invariant mean, then $G$ satisfies a Folner condition. (Hence, amenability and the Folner conditions are equivalent.)

**Proof.** Let $\epsilon > 0$ and $S \subseteq G$ be a finite set. We aim to find finite $F$ so that

$$\max_{x \in S} \frac{|xF \triangle F|}{|F|} < \epsilon.$$

By the existence of an approximate left invariant mean we know there is some $\mu \in P(G) \subseteq \ell^1(G)$ so that $\|x * \mu - \mu\|_1 < \epsilon$. The finite $F$ we need will come from the layer-cake decomposition of $\mu$.

We recall this decomposition and make some observations about it. For $f \in \ell^1(G)$ positive and $r \geq 0$, define

$$F(f,r) = \{ t \in G | f(t) > r \}.$$

Observe that for $f, h \in \ell^1(G)$ positive we have

$$|\chi_{F(f,r)}(t) - \chi_{F(h,r)}(t)| = 1 \iff r \text{ is in between } f(t) \text{ and } h(t).$$

Hence, if $f, h$ are bounded by 1,

$$|f(t) - h(t)| = \int_0^1 |\chi_{F(f,r)} - \chi_{F(h,r)}| dr.$$
Now we apply these observations to \( f = x \ast \mu \) and \( h = \mu \) to obtain

\[
\|x \ast \mu - \mu\|_1 = \sum_{t \in G} |x(t) - \mu(t)| = \sum_{t \in G} \int_0^1 |\chi_{F(x \ast \mu, \tau)}(t) - \chi_{F(\mu, \tau)}(t)| \, dr
\]

\[
= \int_0^1 \sum_{t \in G} |\chi_{F(x \ast \mu, \tau)}(t) - \chi_{F(\mu, \tau)}(t)| \, dr \quad \text{(By Fubini’s Theorem)}
\]

\[
= \int_0^1 \sum_{t \in G} |\chi_{F(\mu, \tau)}(t) - \chi_{F(\mu, r)}(t)| \, dr = \int_0^1 \sum_{t \in G} |\chi_{F(\mu, \tau)}(t) - \chi_{F(\mu, \tau)}(t)| \, dr
\]

\[
= \int_0^1 |x F(\mu, \tau) \nabla F(\mu, \tau)| \, dr
\]

Also notice that \( \int_0^1 |F(\mu, \tau)| \, dr = 1 \). Indeed,

\[
\int_0^1 |F(\mu, \tau)| \, dr = \int_0^1 \sum_{t \in \mathcal{G}} |\chi_{F(\mu, \tau)}| = \sum_{t \in \mathcal{G}} \int_0^1 |\chi_{F(\mu, \tau)}| = \sum_{t \in \mathcal{G}} \int_0^1 |\chi_{\mu(t)}| \, dr
\]

\[
= \sum_{t \in \mathcal{G}} \int_0^{\mu(t)} 1 \, dr = \sum_{t \in \mathcal{G}} \mu(t) = 1
\]

since we assume \( \mu \in P(G) \).

Combining these two gives

\[
\epsilon \int_0^1 |F(\mu, \tau)| \, dr = \epsilon > \|x \ast \mu - \mu\|_1 = \int_0^1 |x F(\mu, \tau) \nabla F(\mu, \tau)| \, dr.
\]

Hence, there is some \( \tau \) so that

\[
|\chi_{F(\mu, \tau)}| - \epsilon |F(\mu, \tau)| < \epsilon \quad \text{for every } x \in \mathcal{G}
\]

for every \( x \in \mathcal{G} \) and hence for every \( x \in S \).

\[
\square
\]

### 2.4 Almost Invariant vectors in \( L^2(G) \)

We consider a characterization of amenability in terms of the Hilbert space \( L^2(G) \). Thus, we can think of it as a representation theoretic characterization.

Recall the **left regular representation** of \( G \) denoted by \( \lambda : G \to B(L^2(G)) \) which sends \( g \in G \) to the translation operator \( \lambda_g \). Explicitly, for \( f \in L^2(G) \), \( \lambda(g)f(h) = f(g^{-1}h) \). In fact, this is a unitary representation since Haar measure is translation invariant.

We need one more definition before giving our characterization.

**Definition 9.** We say the left regular representation of \( L^2(G) \) has **almost invariant vectors** if for every \( \epsilon > 0 \), there is \( f \in L^2(G) \) of unit norm with

\[
\|\lambda_g f - f\|_2 < \epsilon.
\]
Theorem 10. $G$ is amenable if and only if the left regular representation of $L^2(G)$ has almost invariant vectors.

Proof. For the forward direction we use that $G$ being amenable means that it has a Folner sequence $(F_n)$. For every $n \geq 1$, let 

$$f_n = \frac{\chi_{F_n}}{|F_n|^{1/2}}.$$ 

This normalization has that $f_n$ has unit norm. A computation shows 

$$\langle \lambda_g f_n, f_n \rangle = \frac{1}{|F_n|} \| \chi_{g^{-1}F_n} \chi_{F_n} \|_2 = \frac{|g^{-1}F_n \cap F_n|}{|F_n|}.$$ 

As $n$ tends to infinity, this quantity tends to 1 since $(F_n)$ is a Folner sequence. Now, we have that 

$$\| \lambda_g f_n - f_n \|_2^2 = \langle \lambda_g f_n - f_n, \lambda_g f_n - f_n \rangle = 2(1 - \langle \lambda_g f_n, f_n \rangle) \to 0.$$ 

For the backwards direction, we show that $L^2(G)$ having invariant vectors means that $G$ has the approximate left invariant mean property. Indeed, let $\varepsilon = 1/n$ and choose a sequence of almost invariant vectors $(f_n)$ of unit norm. We claim their squares will do the trick: that is, we claim $\| g f_n^2 - f_n^2 \|_1$ tends to 0. Indeed, since 

$$g f_n^2 - f_n^2 = (g f_n + f_n)(g f_n - f_n)$$ 

as functions, then by Cauchy-Schwartz 

$$\| g f_n^2 - f_n^2 \|_1 \leq \| g f_n + f_n \|_2 \cdot \| g f_n - f_n \|_2 \leq 2 \| f_n \|_2 \cdot \| g f_n - f_n \|_2 \to 0$$ 

as claimed. \qed

2.5 Summary of characterizations

We recall the different characterizations of amenability.

Theorem 11. The following are equivalent:

1. $G$ is amenable,
2. $G$ has approximate left invariant means,
3. $G$ satisfies a Folner condition,
4. The left regular representation of $G$ has almost invariant vectors in $L^2(G)$.

3 Amenability and Group $C^*$-algebras

In this section we introduce group algebras. The basic idea is that given a group $G$, we can construct a $C^*$ algebra out of $G$ that captures all the unitary representation theory of $G$. In fact, we construct two such $C^*$-algebras, the reduced and the full $C^*$ algebra. In turns out that amenability can be characterized by the equivalence of these two objects. We give a flavor of the kind of arguments involved by showing that amenability implies the equivalence of these the reduced and full $C^*$ algebras.
3.1 Group $C^*$-algebras

We review some basic notions from Banach algebras for completeness.

**Definition 12.** A Banach algebra is a complex normed algebra $A$ which is complete and satisfies $\|AB\| \leq \|A\|\|B\|$.

A $C^*$-algebra is a Banach algebra with an involution $^*$ such that $\|AA^*\| = \|A\|^2$.

We illustrate these notions with some examples.

**Example 5.** Let $G$ be a discrete group with Haar measure $\mu$. Let $L^1(G) := L^1(G, \mu)$ be the set of integrable functions. We equip $L^1(G)$ with the involution $f^*(t) = \overline{f}(t^{-1})$ and multiplication given by convolution $(f * g)(t) = \sum_{s \in G} f(s)g(s^{-1}t)$. We leave it as an exercise to show that this is a Banach algebra, but not a $C^*$-algebra. (Hint: consider the function $f = \delta_0 + i\delta_1 + \delta_2$.)

Rather than working directly with $L^1(G)$, we often work with the dense subset $CG := \left\{ \sum_{s \in F \subset G} \alpha_s \delta_s : F \subset G \text{ is finite and } \alpha_s \in \mathbb{C} \right\}$.

**Example 6.** Let $C(X)$ be the set of continuous functions $f : X \to \mathbb{C}$ on a compact Hausdorff set $X$. Equip it with the supremum norm, multiplication given by point-wise multiplication, and involution given by $f^*(t) = \overline{f}(t)$. It is not hard to see that this is a $C^*$ algebra.

**Example 7.** Let $B(H)$ be the set of bounded operators on a Hilbert space with the operator norm. It has multiplication given by composition and involution given by the adjoint. This too is a $C^*$-algebra.

We now move on to the construction of the full and reduced algebras. The motivation for these is to understand the (unitary) representation theory of a discrete group $G$. (Recall that a unitary representation of $G$ is just a homomorphism $\pi : G \to U(H) \subset B(H)$ which is strong operator topology continuous and where $U(H)$ is the set of unitary operators on some Hilbert space $H$.) We like to consider $L^1(G)$ because there is a one-to-one correspondence between

$$\left\{ \text{Unitary representations of } G \right\} \leftrightarrow \left\{ \text{non-degenerate representations of } L^1(G) \right\}.$$ 

Recall that a non-degenerate representation is a representation $\pi : G \to B(H)$ such that $\pi(G)H = H$.

Indeed, for the forward direction we can extend a representation of $G$ to $L^1(G)$ by

$$\pi(f) := \sum_{t \in G} f(t)\pi(t)$$

for $f \in L^1(G)$. Also, notice that we are abusing notation by referring to the extension as $\pi$ still. Effectively what we are doing is identifying $G$ inside $CG$ by $s \mapsto \delta_s$ and then extending by linearity. We notice that $\pi(f)$ is indeed a bounded operator by the triangle inequality since

$$\|\pi(f)\| \leq \sum_{t \in G} |f(t)|\|\pi(t)\| = \|f\|_1 < \infty.$$ 

It is not hard to see that the extension to $L^1(G)$ is still a representation. For example, showing $\pi(f * g) = \pi(f)\pi(g)$ is just an application of Fubini’s theorem. Conversely, it is not hard to see that a non-degenerate representation of $L^1(G)$ produces a unitary one on $G$ by restriction.
Example 8. (Left regular representation) Every discrete group $G$ has the left regular representation $\lambda : G \to B(L^2(G))$ which sends $s \in G$ to the translation operator $\lambda_s$. Explicitly, for $g \in L^2(G)$, $\lambda(s)g(t) = g(s^{-1}t)$. In fact, this is a unitary representation since Haar measure is translation invariant.

By our construction above, we can extend $\lambda$ from $G$ to $L^1(G)$. We do this by interpreting the definition of the extension as the operator defined in the inner product, that is.

$$\langle \lambda(f)g, h \rangle = \sum_{t \in G} f(t) \langle \lambda_s g, h \rangle.$$ 

This interpretation of $\lambda(f)$ works for groups that are not necessarily discrete and so this gives a preview to how the general locally compact case is handled. We compute the extension via this interpretation:

$$\langle \lambda(f)g, h \rangle = \sum_{s \in G} f(s) \langle \lambda_s g, h \rangle = \sum_{s \in G} f(s) \left( \sum_{t \in G} g(s^{-1}t) h(t) \right)$$

$$= \sum_{s \in G} \sum_{t \in G} f(s) g(s^{-1}t) h(t)$$

$$= \sum_{t \in G} \left( \sum_{s \in G} f(s) g(s^{-1}t) \right) h(t) \quad \text{(By Fubini’s theorem)}$$

$$= \sum_{t \in G} \left( \sum_{s \in G} f(s) g(s^{-1}t) \right) h(t) = \sum_{t \in G} (f*g)(t) h(t)$$

$$= \langle f * g, h \rangle.$$

Hence, the extension $\lambda : L^1(G) \to B(L^2(G))$ is

$$\lambda(f)g = f * g.$$ 

This holds whether or not $G$ is discrete and we leave it to the reader to verify this. (Hint: replace all the sums by integrals.)

If we take the operator closure of the image of $\lambda : L^1(G) \to B(L^2(G))$, we obtain a $\mathcal{C}^*$-algebra from $G$. We state this formally.

Definition 13. The reduced $\mathcal{C}^*$-algebra is defined as the operator closure of the image of $\lambda : L^1(G) \to B(L^2(G))$,

$$\mathcal{C}^*_r(G) := \overline{\lambda(L^1(G))}_{\| \cdot \|_{op}}.$$ 

A priori, there is nothing special about the left regular representation and we can generalize this construction to obtain the full $\mathcal{C}^*$-algebra $\mathcal{C}^*(G)$: We consider the direct sum of all unitary representations $\pi_u$ up to unitary equivalence, $\pi = \oplus_u \pi_u$. Define the full $\mathcal{C}^*$-algebra to be

$$\mathcal{C}^*(G) := \overline{\pi(L^1(G))}_{\| \cdot \|_{op}}.$$ 

Are these really $\mathcal{C}^*$-algebras? What is the involution? The point here is that we are a subset of $B(H)$ so that we already come equipped with the adjoint as an involution. The only thing to show is that the adjoint is closed i.e. $\pi(f)^* = \pi(f^*)$. A representation that commutes with the involution $^*$ is said to be a $^*$-representation.
To define the full $C^*$-algebra, we can equivalently just consider the norm
$$\|f\|_{C^*(G)} := \sup\{\|\pi(f)\| : \pi \text{ is a }^*\text{-representation of } L^1(G) \}$$
and define
$$C^*(G) := \overline{L^1(G)}^{\|\cdot\|_{C^*(G)}}.$$

We state the main theorem of this section.

**Theorem 14.** Let $G$ be a discrete group. Then $G$ amenable if and only if $C^*(G) = C^r(G)$.

The forward direction is the more difficult one and we focus on that at first. Notice we automatically have the surjection $\lambda: L^1(G) \to \lambda(L^1(G)) = C^r(G)$. So that, by extending this representation, we also have the surjection $\lambda: L^1(G)^{\|\cdot\|_{C^*(G)}} = C^*(G) \to C^r(G)$. Hence, to prove the equivalence of the $C^*$-algebras we aim to show this map is injective by showing it is an isometry.

### 3.2 Positive Definite Functions

In order to prove our main theorem, we have to venture into the world of positive definite functions.

**Definition 15.** A function $\varphi: G \to \mathbb{C}$ is positive definite if
$$\sum_{i=1}^n \sum_{j=1}^n \alpha_i \overline{\alpha}_j \varphi(s_j^{-1}s_i) \geq 0$$
for all $n \geq 1$, $\alpha_i \in \mathbb{C}$, $s_i \in G$.

We bring positive definite functions into the picture via the following proposition.

**Proposition 16.** There is a one-to-one correspondence between

$$PD(G) := \left\{ \varphi: G \to \mathbb{C} : \varphi \text{ is positive definite and } \varphi(e) = 1 \right\} \leftrightarrow \left\{ \text{States on } C^*(G) \right\}.$$

When we say state here we are referring to the more general notion of a mean i.e. a positive linear functional of norm 1 on a $C^*$-algebra. Positive means $\Phi(f) \geq 0$ whenever $f \geq 0$ i.e. whenever $f$ is a positive element in the $C^*$-algebra.

**Proof.** Given a state $\Phi$, define $\varphi(s) := \Phi(\delta_s)$. Then clearly $\varphi(e) = 1$ and if $f = \sum_{i=1}^n \alpha_i \delta_{s_i}$, then
$$\sum_{i=1}^n \sum_{j=1}^n \alpha_i \overline{\alpha}_j \varphi(s_j^{-1}s_i) = \Phi(f^* * f) \geq 0,$$
since $f^* * f \geq 0$ and $\Phi$ is a mean/state.

On the other hand, if $\phi$ is positive definite, then define a functional on $\mathbb{C}G$ by
$$\Phi\left( \sum_{i=1}^n \alpha_i \delta_{s_i} \right) := \sum_{i=1}^n \alpha_i \varphi(s_i).$$
So, if \( f = \sum_{i=1}^{n} \alpha_i \delta_{s_i} \), then
\[
\Phi(f^* f) = \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \bar{\alpha}_j \varphi(s_i^{-1}s_j) \geq 0,
\]
since \( \varphi \) is a positive definite.

The upshot is the following: To prove our main theorem we want to show that \( \lambda : C^*(G) \rightarrow C^*_\epsilon(G) \) is an isometric \(*\)-isomorphism. Well, clearly it is a surjective \(*\)-homomorphism so by virtue of it being a map between two \( C^* \) algebras we have
\[
\|\lambda(f)\|_{op} \leq \|f\|_{C^*(G)}.
\]

So to finish the argument, all we have to do is show the reverse inequality of the norms. Basic functional analysis tells us that we can recover the norm of an element \( f \) in a \( C^* \)-algebra by knowing how linear functionals eat \( f \). Actually for positive elements, it suffices to look at how means eat elements and our proposition says means are just the same as positive definite functions. Consequently, for the element \( f = \sum_{i=1}^{n} \alpha_i \delta_{s_i} \) that is in the dense subset \( C^r(G) \), we have
\[
\|f\|_{C^r(G)} = \|f^* f\|_{C^r(G)}^{1/2} = \sup_{\Phi \text{ a state}} \Phi(f^* f)^{1/2} = \sup_{\varphi \in PD(G)} \left( \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \bar{\alpha}_j \varphi(s_i^{-1}s_j) \right)^{1/2}.
\]

We often denote the term on the far right as \( \sup_{\varphi \in PD(G)} \varphi(f^* f)^{1/2} \). Now that we have the connection to positive definite functions and how they fit into the discussion, we develop some tools for positive definite functions.

**Lemma 17.** If \( \varphi_1 \) and \( \varphi_2 \) belong to \( PD(G) \), then \( \varphi_1 \varphi_2 \in PD(G) \).

The following lemma is a special case of the GNS construction that says all states \( \varphi \) (which are just positive definite functions by our correspondence) have the form \( \varphi(s) = \langle \pi(s)x, x \rangle \) for some representation \((\pi, H)\) and \( x \in H \). It says when we have the additional hypothesis of finite support, then we can pick a nice representation (the left regular representation) for our state to look like.

**Lemma 18.** If \( \varphi \in PD(G) \) has finite support, then there is a unit vector \( x \in L^2(G) \) with \( \varphi(s) = \langle \lambda(s)x, x \rangle \), where \( \lambda \) is the left regular representation of \( G \).

Proof. Consider the convolution operator on \( CG \) given by \( Tf = f \ast \varphi \). By Young’s inequality,
\[
\|Tf\| = \|f \ast \varphi\| \leq \|f\|_2 \|\varphi\|_1.
\]

Hence, \( T \) is a bounded operator so that it admits an extension to \( L^2(G) \). We will show that \( T \) is a positive operator so that it has a square root. Consider the following computation to show that \( T \) is positive: let \( f = \sum_{i=1}^{n} \alpha_i \delta_{s_i} \)
\[
\langle Tf, f \rangle = \sum_{s \in G} \sum_{t \in G} f(t) \varphi(t^{-1}s) \bar{f}(s) = \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \bar{\alpha}_j \varphi(s_i^{-1}s_j) \geq 0
\]
since \( \varphi \) is positive definite. Hence \( T \) has a square root, \( T^{1/2} \).

Note \( \lambda(s)T = T\lambda(s) \). Indeed,
\[
\lambda(s)Tf = \delta_s * (f \ast \varphi) = (\delta_s * f) \ast \varphi = T(\delta_s * f) = T\lambda(s)f.
\]

We claim that \( x = T^{1/2}(s) \) does the trick:
\[
(\lambda(s)x, x) = (\lambda(s)T^{1/2}(s), T^{1/2}(s)) = (\delta_s, T\delta_s) = (\delta_s, \varphi) = \varphi(s).
\]

Lastly, \( ||x||_2 = \varphi(e) = 1 \) and this completes the proof.

\[\square\]

3.3 Proof of Main Theorem

We are now ready to prove the equivalence of the full and reduced \( C^* \)-algebra of a discrete amenable group \( G \).

**Proof.** Suppose \( G \) is amenable. Then there is an approximate left invariant mean \( (g_n) \subseteq P(G) \) with
\[
\|\delta_s * g_n - g_n\|_1 \to 0,
\]
for every \( s \in G \). Normalizing we can make them satisfy \( \|g_n\|_1 = 1 \).

Take \( h_n = g_n^{1/2} \) so that \( (h_n) \) has unit \( L^2(G) \)-norm.

Consider the trivial equality \( |a - b|^2 = |a^2 - b^2| \) for \( a, b \geq 0 \). We use this on the positive \( (h_n) \): for every \( s \in G \),
\[
\lim_n \|\lambda(s)h_n - h_n\|_2^2 = \lim_n \sum_{t \in G} |\lambda(s)h_n(t) - h_n(t)|^2 = \lim_n \sum_{t \in G} |h_n(s^{-1}t) - h_n(t)|^2 \\
\leq \lim_n \sum_{t \in G} |g_n(s^{-1}t) - g_n(t)| = \lim_n \|\delta_s * g_n - g_n\|_1 \to 0.
\]

Now, since \( \langle \lambda(s)h_n, h_n \rangle \) is real, then
\[
\lim_n \langle \lambda(s)h_n, h_n \rangle = \lim_n \langle \lambda(s)h_n, h_n \rangle + \frac{1}{2} \lim_n \|\lambda(s)h_n - h_n\|_2^2 = \lim_n \lambda(s)h_n + \frac{1}{2} \langle \lambda(s)h_n - h_n, \lambda(s)h_n - h_n \rangle = \frac{1}{2} \lim_n (\|\lambda(s)h_n\|_2^2 + \|h_n\|_2^2) \\
= \lim_n \|h_n\|_2^2 = 1.
\]

Why is this important? Set \( \varphi(s) = \langle \lambda(s)h_n, h_n \rangle \). Then this last calculation shows that we obtain a sequence \( \varphi_n \in PD(G) \) with finite support and such that \( \varphi_n \to 1 \) point-wise.

This will allow us to finish the proof. Let \( \varphi \in PD(G) \). Then \( \varphi\varphi_n \in PD(G) \) and it continues to have finite support. So by our lemma, there is \( (x_n) \subseteq L^2(G) \) with \( \varphi\varphi_n(s) = \langle \lambda(s)x_n, x_n \rangle \) for all \( s \in G \). Hence, for \( f = \sum_{i=1}^n \alpha_i \delta_{s_i} \in \mathbb{C}G \),
\[
\varphi(f^* \ast f) = \sum_{s \in G} \varphi(s)(f^* \ast f)(s) = \lim_n \sum_{s \in G} \varphi_n(s)\varphi(s)(f^* \ast f)(s) \\
= \lim_n \langle \lambda(f^* \ast f)x_n, x_n \rangle \leq \|\lambda(f^* \ast f)\| = \|\lambda(f)\|_2^2.
\]
where on the second to last line we used Cauchy-Schwartz. Taking the supremum over \( \varphi \in PD(G) \), we obtain

\[
\| \lambda(f) \|_{op} \geq \| f \|_{C^*(G)}
\]

which proves our theorem.

\[\square\]

The converse is much easier.

**Proposition 19.** If \( G \) is discrete and has \( C^*_r(G) = C^*(G) \), then \( G \) is amenable.

**Proof.** Let \( \varepsilon : G \to \mathbb{C} \) be the trivial representation i.e. \( \varepsilon(s) = 1 \) for every \( s \in G \). Then it clearly extends to \( L^1(G) \) and hence to \( C^*(G) \). Since \( C^*_r(G) = C^*(G) \), our representation is just \( \varepsilon : C^*_r(G) \subseteq B(L^2(G)) \to \mathbb{C} \) which is a *-homomorphism. After extending it to \( L^2(G) \) (By Hahn-Banach for states), we can just regard it as a mean (as in section 1) since \( L^\infty(G) \subseteq B(L^2(G)) \). Finally, notice that we have (as an action on \( L^2(G) \))

\[
\lambda_s f \lambda^*_s = \lambda_s f
\]

where e.g. \((\lambda_s f)(t) = \delta_s * (f(t)g(t))\). Hence, \( \varepsilon : L^\infty(G) \to \mathbb{C} \) satisfies

\[
\varepsilon(\lambda_s f) = \varepsilon(\lambda_s f \lambda^*_s) = \varepsilon(\lambda_s) \varepsilon(f) \varepsilon(\lambda^*_s)
\]

\[
= \varepsilon(\lambda_s) \varepsilon(f) \varepsilon(\lambda_s) = \varepsilon(f)
\]

since \( \lambda_s \) is a unitary operator and \( \varepsilon \) is a *-morphism. In particular, \( \varepsilon \) is an invariant mean so that \( G \) is amenable.

\[\square\]