The WP and FP problem sets you work on for our class have different purposes, and correspondingly different requirements for the way you write them up. In WP work and the associated classwork, you are getting acquainted with the material, brainstorming, drafting solutions. The FP sets have the goals of showing how well you have mastered recent material, and also of working on your mathematical exposition skills.

The requirements for write-ups are described in the files WPguidelines and FPguidelines at the Canvas site, but no examples are given. Here is one problem written up in four ways.

1. The first would get little or no credit as a WP solution, and none as a FP solution.
2. The second is the WP solution of a student who doesn't understand the the problem much at all, but made and recorded a reasonable attempt to work on it. This would get full credit as a WP solution, but only little or no credit as an FP solution.
3. The third is a WP solution that shows the student understood the problem almost completely, so of course is full credit as a WP solution. As an FP solution, it would lose at least one point for not being written up formally, with complete sentences, all notation and reasoning explained, etc.
4. The last solution is a full credit FP solution.

The Problem, (an expanded version of §2, p.21, Exercise 2(g)) Let $f : A \to B$ and let $A_i \subset A$ and $B_i \subset B$ for $i = 0, 1$.

(i) Prove that $f(A_0 \cap A_1) \subset f(A_0) \cap f(A_1)$.

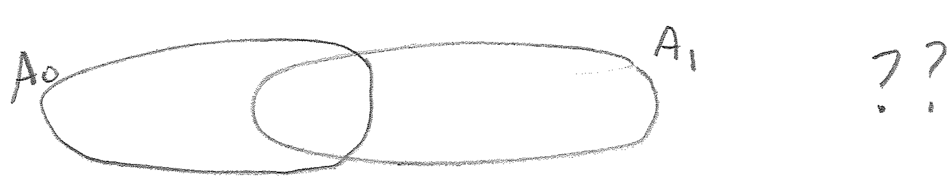
(ii) Show that equality holds if f is injective.

(iii) Give an example in which $f(A_0 \cap A_1)$ and $f(A_0) \cap f(A_1)$ are not equal.

(iv) Determine conditions you can add to make an "if and only if" theorem. State the theorem and prove it.

If you didn't work on this in the first class and have time now, think a little about how to solve this before reading the solutions.

First solution. Little or no credit, even for WP.

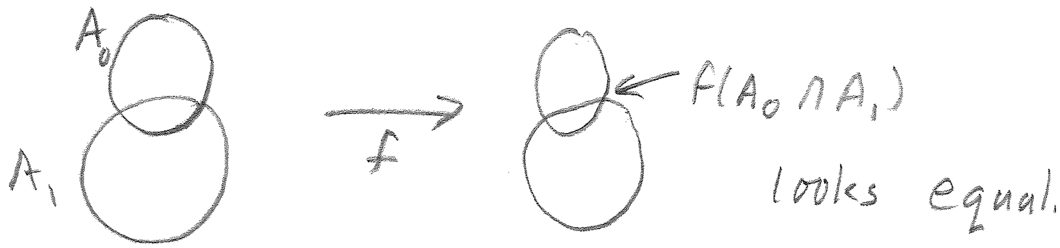


Injective f: only one a per $f(a)$.

I don't know how to do this.

Remark: The comment here about f being injective isn't very useful, because it isn't in a form that's precise enough to be the definition. See definition stated in the third solution.

Second solution. Full credit as WP, because student shows a serious attempt to work on the solution, even though it is clear the student doesn't understand very much. Little credit as a FP solution.



(ii) Not injective example: $f(x) = x^2$

$A_0 = [0, 2]$, $A_1 = [1, 3]$

$f(\ )$: $[0\ 4]$ $[1, 9]$

$\cap = [1, 2] \to [1, 4]$ still equal?

injective: $f(a) = b = y = f(x) \Rightarrow a = x$

Don't know what to try next.

Third solution. Very good full credit WP solution. Even if it had a solution for the last part, would not be full credit as a FP solution because it's not a formal writeup.

(i) $b \in f(A_0 \cap A_1)$, so $= f(a), a \in A_0 \cap A_1$

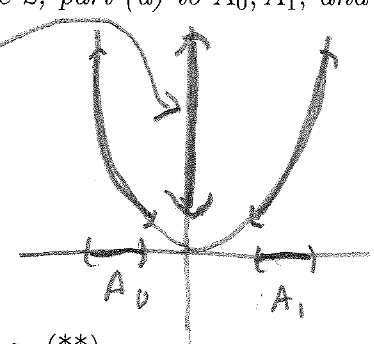
$a \in A_0, A_1$, so $b = f(a) \in f(A_0)$ and $f(A_1)$ and so also in $\cap$.

Remark: For a different proof, apply Exercise 2, part (a) to A_0, A_1, and $A_0 \cap A_1$.

(ii) $f(x) = x^2$.

$A_0 \cap A_1 = \emptyset$,

$f(A_0) \cap f(A_1) \neq \emptyset$



(iii) f injective means $f(a_0) = f(a_1) \Rightarrow a_0 = a_1$ (**)

$b \in f(A_0) \cap f(A_1) \Rightarrow b = f(a_0)$ for $a_0 \in f(A_0)$ and $b = f(a_1)$ for $a_1 \in f(A_1)$. So $b \in f(A_0) \cap f(A_1)$

(iv) wts for any $a_0, a_1 \in A$, (**) holds.

Don't see why LHS should imply this unless I just assume injective???
Out of time!

Fourth solution. Full credit FP solution.

Throughout the problem, $f: A \to B$ and $A_i \subset A$, $B_i \subset B$ for $i = 0,1$.

(i) $f(A_0 \cap A_1) \subset f(A_0) \cap f(A_1)$

Pf. Suppose $b \in f(A_0 \cap A_1)$, so $b = f(a)$ for some $a \in A_0 \cap A_1$. That implies $a \in A_0$, so $b = f(a) \in f(A_0)$, and similarly $b \in f(A_1)$. Thus $b \in f(A_0) \cap f(A_1)$. ■

(ii) Equality fails for $f(x) = x^2$, $A_0 = (-\infty, 0)$, $A_1 = (0, \infty)$. We see $A_0 \cap A_1 = \emptyset$, so $f(A_0 \cap A_1) = \emptyset \neq f(A_0) \cap f(A_1) = (0, \infty)$.

(iii) By definition, a function f is injective if

$$f(a_0) = f(a_1) \implies a_0 = a_1. \qquad (**)$$

Suppose $b \in f(A_0) \cap f(A_1)$. This means $b = f(a_0)$ for some $a_0 \in A_0$, and also $b = f(a_1)$ for some $a_1 \in A_1$. Then we have $f(a_0) = f(a_1)$, so by $(**)$, $a_0 = a_1$. Thus $a_0 = a_1 \in A_0 \cap A_1$, and so $b \in f(A_0 \cap A_1)$. ■

(iv) Theorem. With notation as given above, suppose that for all choices of the subsets A_0 and A_1,

$$f(A_0 \cap A_1) = f(A_0) \cap f(A_1).$$

Then f is injective.

Pf. Suppose a_0 and a_1 are elements of A and $b = f(a_0) = f(a_1)$. Let $A_0 = \{a_0\}$ and $A_1 = \{a_1\}$, so $f(A_0) \cap f(A_1) = \{b\}$ is not empty. Our hypothesis then says $f(A_0 \cap A_1) = \{f(b)\}$ also is not empty. Because A_0 and A_1 are singleton sets, it must be that $a_0 = a_1$. So f satisfies $(**)$ and is injective. ■