Solstice Mathematics:
Solar Declination and Daylight Length

Notes

Andrew D. Loveless

1. Big Picture

These notes develop a mathematically precise but accessible model for:

- why the *longest* and *shortest* days of the year occur, and
- how to compute the *daylight duration* for a given latitude.

We work under a simplified but powerful geometric model:

- Earth is a sphere.
- The Sun is very far away, so sunlight rays are parallel.
- Earth orbits the Sun in a circle at constant angular speed.
- Earth's rotation axis is fixed in space and tilted by the obliquity $\varepsilon \approx 23.44^\circ$ relative to the ecliptic plane.[1]

This model is accurate enough to explain solstices and produces daylight values that closely match published tables, especially once small refinements (refraction, elliptical orbit, solar disk size) are acknowledged.[2]

2. Projections and Dot Products

Two vector facts drive nearly everything that follows.

Dot product and angle

For vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$,

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \, \|\mathbf{b}\| \cos\theta,$$

where θ is the angle between them. If $\|\mathbf{b}\| = 1$, then $\mathbf{a} \cdot \mathbf{b}$ is the signed component of $\mathbf{a}$ in the $\mathbf{b}$-direction.

[1] For the current value of the obliquity, see wikipedia.org/wiki/Axial_tilt.
[2] NOAA provides solar calculation references and tables: noaa.gov/sunsetsunrise-calculator.

Projection formula

The vector projection of $\mathbf{a}$ onto $\mathbf{b}$ is

$$\operatorname{proj}_{\mathbf{b}}(\mathbf{a}) = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{b}\|^2}\,\mathbf{b},$$

and the scalar component is

$$\operatorname{comp}_{\mathbf{b}}(\mathbf{a}) = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{b}\|}.$$

These ideas appear twice in our model:

- projecting the Sun's direction onto Earth's rotation axis (solar declination),
- determining whether a point on Earth faces the Sun (day vs. night).

3. Solar Declination from a Circular Orbit

Definition

The **solar declination** δ is the angle between the direction to the Sun and Earth's equatorial plane. Equivalently,

$$\sin\delta(t) = \mathbf{a}\cdot\mathbf{s}(t),$$

where $\mathbf{a}$ is Earth's axis unit vector and $\mathbf{s}(t)$ is the Sun-direction unit vector.

Coordinate setup

Let the ecliptic plane be the xy-plane. Earth's orbital position is

$$\mathbf{r}(t) = \big(\cos\theta(t),\, \sin\theta(t),\, 0\big), \qquad \theta(t) = \omega t + \theta_0, \qquad \omega = \frac{2\pi}{T_y},$$

with $T_y \approx 365.24$ days.

The Sun direction from Earth is

$$\mathbf{s}(t) = -\mathbf{r}(t).$$

Choose coordinates so Earth's rotation axis is obtained by tilting the ecliptic normal by ε about the y-axis:

$$\mathbf{a} = \big(\sin\varepsilon,\, 0,\, \cos\varepsilon\big).$$

Computing the declination

Using $\sin\delta = \mathbf{a}\cdot\mathbf{s}$,

$$\sin\delta(t) = -\sin\varepsilon\cos(\omega t + \theta_0),$$

so that

$$\delta(t) = \arcsin\Big(-\sin\varepsilon\cos(\omega t + \theta_0)\Big).$$

Why declination is nearly sinusoidal

Since $\arcsin(x) \approx x$ for moderate $|x|$, a common approximation is

$$\delta(t) \approx \varepsilon\sin(\omega t + \phi),$$

with ϕ chosen so $\delta = 0$ at an equinox.

4. Daylight Length as a Circle-Cut-by-a-Line Problem

Geometric idea

Fix a latitude φ. As Earth rotates, the observer traces a latitude circle. Daylight occurs when the observer lies on the sunlit hemisphere. This reduces the problem to determining what fraction of a circle lies on one side of a line.

Earth-centered coordinates

A point at latitude φ has position

$$\mathbf{r}(H) = \Big(R_E \cos\varphi \cos H,\ R_E \cos\varphi \sin H,\ R_E \sin\varphi\Big),$$

where H (hour angle) runs from 0 to 2π over one day.

Assume the Sun direction is

$$\mathbf{s} = (\cos\delta,\, 0,\, \sin\delta).$$

Daylight condition

A point is in daylight when

$$\mathbf{r}(H) \cdot \mathbf{s} > 0.$$

Solving the equality gives

$$\cos H_0 = -\tan\varphi \tan\delta.$$

Daylight duration

Daylight occurs for $-H_0 \le H \le H_0$. Thus the daylight duration is

$$D(\varphi, \delta) = 24 \cdot \frac{1}{\pi} \arccos\big(-\tan\varphi \tan\delta\big).$$

Polar day and polar night

If $|-\tan\varphi \tan\delta| > 1$, then:

$$\begin{cases} D = 24 & \text{(continuous daylight)}, \\ D = 0 & \text{(continuous darkness)}. \end{cases}$$

Real-world examples

- **Northern Hemisphere:** Utqiagvik (Barrow), Alaska experiences 24-hour daylight in summer.
- **Southern Hemisphere:** McMurdo Station, Antarctica experiences 24-hour daylight during austral summer.

5. Solstices and Extreme Day Lengths

In this model, solstices occur when

$$\delta = \pm\varepsilon.$$

For a given latitude φ,

$$D_{\max}(\varphi) = \max\{D(\varphi, +\varepsilon), D(\varphi, -\varepsilon)\}, \quad D_{\min}(\varphi) = \min\{D(\varphi, +\varepsilon), D(\varphi, -\varepsilon)\}.$$

6. Computed Daylight Extremes for Selected Locations

Place	Latitude	Max Light	Solstice	Min Light	Solstice
Seattle, WA	47.6	15:47	June	08:13	Dec.
Manzanita, OR	45.7	15:31	June	08:29	Dec.
Disneyland, CA	33.8	14:15	June	09:45	Dec.
Kahului (Maui)	20.8	13:16	June	10:44	Dec.
Anchorage, AK	61.2	18:57	June	05:03	Dec.
Sydney, Australia	−33.9	14:15	Dec.	09:45	June

7. Why the Sinusoidal Approximation Works

The Taylor series

$$\sin x = x - \frac{x^3}{6} + \cdots$$

implies the error bound

$$|\sin x - x| \leq \frac{|x|^3}{6}.$$

With $\varepsilon \approx 0.409$ rad,

$$|\sin\varepsilon - \varepsilon| \leq 0.0114,$$

which is under 3%. Thus the sinusoidal declination model captures the dominant seasonal behavior.

8. Model Validation and Limitations

Comparisons with NOAA sunrise/sunset tables show close agreement. Remaining discrepancies arise from:

- atmospheric refraction,
- finite solar disk size,
- Earth's slightly elliptical orbit,
- the equation of time.

Appendix: Desmos Visualizations

Interactive visualizations accompanying these notes made by the Dr. Andy Loveless:

- **3D Solstice Model:**
  https://www.desmos.com/3d/e9fitb1xcg
- **2D Daylight Circle Model:**
  https://www.desmos.com/calculator/vuhxn5k2ul