1. (10 pts) For both parts below, give the general solution:

(a) $y'' + 2y' + y = 3t^2 - 1$

$r^2 + 2r + 1 = 0 \Rightarrow (r+1)^2 = 0 \Rightarrow r = -1$

$c_1 e^{-t} + c_2 t e^{-t}$

$Y(t) = At^2 + Bt + C, \quad Y'(t) = 2At + B, \quad Y''(t) = 2A$

$(2A) + 2(2At + B) + (At^2 + Bt + C) \stackrel{?}{=} 3t^2 - 1$

$At^2 + (4A + B)t + (2A + 2B + C) = 3t^2 - 1$

$A = 3$

$4A + B = 0 \Rightarrow B = -4A = -12$

$2A + 2B + C = -1 \Rightarrow C = -1 - 2A - 2B = -1 - 6 + 24 = 17$

$$\boxed{y(t) = c_1 e^{-t} + c_2 t e^{-t} + 3t^2 - 12t + 17}$$

(b) $y'' - 4y = 5 + 3e^{2t}$.

$r^2 - 4 = 0 \Rightarrow r = \pm 2$

$c_1 e^{-2t} + c_2 e^{2t}$

$Y(t) = A + Bte^{2t}, \quad Y'(t) = Be^{2t} + 2Bte^{2t}, \quad Y''(t) = 2Be^{2t} + 2Be^{2t} + 4Bte^{2t}$
$= 4Be^{2t} + 4Bte^{2t}$

$(4Be^{2t} + 4Bte^{2t}) - 4(A + \cancel{Bte^{2t}}) \stackrel{?}{=} 5 + 3e^{2t}$

$-4A + 4Bte^{2t} \stackrel{?}{=} 5 + 3e^{2t}$

$-4A = 5 \Rightarrow A = -5/4$

$4B = 3 \Rightarrow B = 3/4$

$$\boxed{y(t) = c_1 e^{-2t} + c_2 e^{2t} - \frac{5}{4} + \frac{3}{4} t e^{2t}}$$

2. (10 pts) **For ALL parts**, assume the mass-spring system has a mass of $m = 2$ kg, a spring constant $k = 5$ N/m, and NO external forcing. Thus, $2u'' + \gamma u' + 5u = 0$. Include UNITS in your final answers.

(a) Assume NO damping and the initial conditions $u(0) = 0.5$ m and $u'(0) = 1$ m/s. Find the solution (find all constants).

$$2u'' + 5u = 0 \Rightarrow 2r^2 + 5 = 0 \Rightarrow r = \pm\sqrt{\tfrac{5}{2}}\, i$$

$$u(t) = c_1 \cos\left(\sqrt{\tfrac{5}{2}}\, t\right) + c_2 \sin\left(\sqrt{\tfrac{5}{2}}\, t\right)$$

$$u'(t) = -\sqrt{\tfrac{5}{2}}\, c_1 \sin\left(\sqrt{\tfrac{5}{2}}\, t\right) + \sqrt{\tfrac{5}{2}}\, c_2 \cos\left(\sqrt{\tfrac{5}{2}}\, t\right)$$

$$u(0) = c_1 \overset{?}{=} 0.5$$

$$u'(0) = \sqrt{\tfrac{5}{2}}\, c_2 \overset{?}{=} 1 \Rightarrow c_2 = \sqrt{\tfrac{2}{5}}$$

$$\boxed{u(t) = 0.5 \cos\left(\sqrt{\tfrac{5}{2}}\, t\right) + \sqrt{\tfrac{2}{5}} \sin\left(\sqrt{\tfrac{5}{2}}\, t\right)}$$

(b) Assume there is damping with $\gamma = 2$ N/(m/s). The solution exhibits vibrations (with decreasing amplitude). What is the quasi-period?

$$2r^2 + 2r + 5 = 0$$

$$r = \frac{-2}{2(2)} \pm \frac{1}{2(2)}\sqrt{4-40}$$

$$r = -\tfrac{1}{2} \pm \tfrac{1}{4} 6i$$

$$r = -\tfrac{1}{2} \pm \tfrac{3}{2} i$$

$$\lambda = -\tfrac{1}{2}, \ \mu = \tfrac{3}{2}$$

$$\text{quasi-period} = \frac{2\pi}{3/2} = \boxed{\frac{4\pi}{3} \text{ seconds}}$$

(c) Give the smallest value of γ for which the solution will NOT exhibit vibrations.

$$\gamma = 2\sqrt{mk} = 2\sqrt{2 \cdot 5} = \boxed{2\sqrt{10}\ \frac{N}{m/s}}$$

3. (10 pts) **For ALL parts**, assume the mass-spring system has a mass of $m = 2$ kg, a spring constant $k = 5$ N/m, **and** an external forcing of the form $F(t) = F_0\cos(\omega t)$ Newtons. Thus, $2u'' + \gamma u' + 5u = F_0\cos(\omega t)$.

(a) Assume there is NO damping. What particular value of ω will lead to vibrations with increasing and unbounded amplitude?

resonance

$$\omega = \omega_0 = \boxed{\sqrt{\tfrac{5}{2}}\ \frac{\text{rad}}{\text{sec}}}$$

(b) Assume there is damping with $\gamma = 2$ N/(m/s) and $u(0) = 0$ m and $u'(0) = 0$ m/s. Also assume $F(t) = 39\cos(t)$ N. You are told the solution takes the form:

$$u(t) = c_1 e^{\lambda t}\cos(\mu t) + c_2 e^{\lambda t}\sin(\mu t) + 9\cos(\omega t) + 6\sin(\omega t).$$

- What are the values of λ, μ, ω, c_1, and c_2? (You only have to give units for μ and ω).

$\boxed{\lambda = -\tfrac{1}{2}}$, $\boxed{\mu = \tfrac{3}{2}\ \frac{\text{rad}}{\text{sec}}}$ ← see previous page

$\boxed{\omega = 1\ \frac{\text{rad}}{\text{sec}}}$ ← same as forcing frequency

$u(0) = 0 \Rightarrow c_1 + 9 = 0 \Rightarrow \boxed{c_1 = -9}$

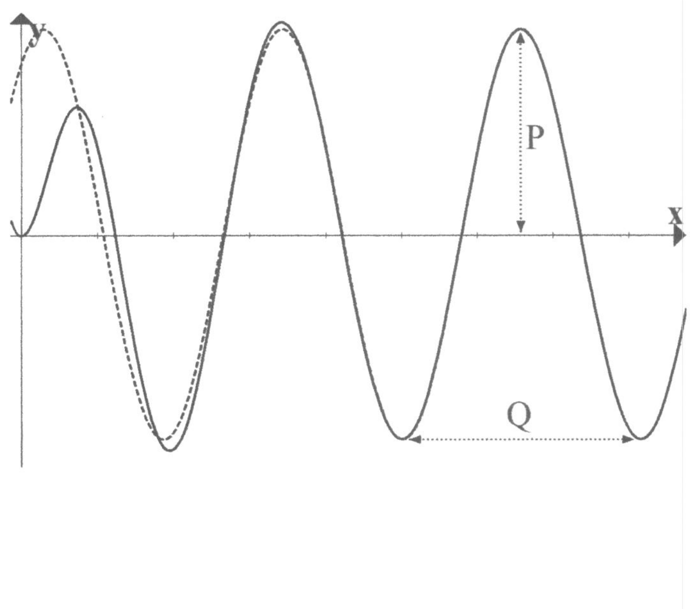
$u'(t) = \lambda c_1 e^{\lambda t}\cos(\mu t) - \mu c_1 e^{\lambda t}\sin(\mu t) + \lambda c_2 e^{\lambda t}\sin(\mu t) + \mu c_2 e^{\lambda t}\cos(\mu t)$
$- 9\omega\sin(\omega t) + 6\omega\cos(\omega t)$

$u'(0) = 0 \Rightarrow \lambda c_1 + \mu c_2 + 6\omega = 0$

$\Rightarrow c_2 = \dfrac{-6\omega - \lambda c_1}{\mu} = \dfrac{-6 - (-\tfrac{1}{2})(-9)}{3/2} = \dfrac{-12 - 9}{3} = -7$

$\boxed{c_2 = -7}$

- The graph of the full solution (solid) and the steady state solution (dotted) are given below. Find the indicated lengths P and Q. (include units!)



$P = \sqrt{9^2 + 6^2}$

$= \sqrt{81 + 36}$ ← $3\sqrt{13}$

$\boxed{P = \sqrt{117}\ \text{meters}}$

$\boxed{Q = \dfrac{2\pi}{\omega} = \dfrac{2\pi}{1} = 2\pi\ \text{seconds}}$

4. (10 pts) Consider the model $u'' + 4u' + 3u = 0$ with $u(0) = 0.3$ m, $u'(0) = -1$ m/s.

(a) What can we say about this system? (**Circle one**):
Critically Damped OR Overdamped OR Exhibits Vibrations.

$\gamma = 4 > 2\sqrt{mk} = 2\sqrt{3}$

(b) Solve for $u(t)$. (find all constants)

$r^2 + 4r + 3 = 0 \Rightarrow (r+3)(r+1) = 0 \Rightarrow r_1 = -1,\ r_2 = -3$

$u(t) = c_1 e^{-t} + c_2 e^{-3t}$

$u'(t) = -c_1 e^{-t} - 3c_2 e^{-3t}$

$u(0) = 0.3 \Rightarrow c_1 + c_2 = 0.3$

$u'(0) = -1 \Rightarrow -c_1 - 3c_2 = -1$

$-2c_2 = -0.7 \Rightarrow c_2 = 0.35$

$c_1 = 0.3 - c_2 = -0.05$

$$\boxed{u(t) = -0.05e^{-t} + 0.35e^{-3t}}$$

(c) Find the one, and only, time the mass will be at the equilibrium position (*i.e.* when $u(t) = 0$).

$-0.05e^{-t} + 0.35e^{-3t} \overset{?}{=} 0$

$0.35e^{-3t} \overset{?}{=} 0.05e^{-t}$

$7 = e^{2t}$

$\ln(7) = 2t$

$$\boxed{t = \tfrac{1}{2}\ln(7) \text{ seconds}}$$

5. (10 pts) (The two parts below are not related)

(a) A 3 kg object stretches a spring 10 cm beyond its natural length (and is at rest). The damping force is 5 N when the upward velocity is 6 m/s. There is no external forcing. Initially, the mass is pushed upward 5 cm and given an initial downward velocity of 20 cm/s. Set up the differential equation AND initial conditions for the displacement $u(t)$. Watch the units! (DO NOT SOLVE)

$\boxed{M=3}$ $L=0.1\,m$ $mg-kL=0 \Rightarrow \boxed{k=\frac{mg}{L} = \frac{3\cdot 9.8}{0.1} = 294 \frac{N}{m}}$

$F_d = -\gamma u' \Rightarrow 5 = -\gamma(-6) \Rightarrow \boxed{\gamma = 5/6 \frac{N}{m/s}}$

$$\boxed{3u'' + \frac{5}{6}u' + 294u = 0 \qquad u(0) = -0.05 \quad u'(0) = 0.2}$$

(b) The function $y_1(t) = t^2$ is one solution to the homogeneous equation $t^2y'' - 2y = 0$. Use reduction of order to find the general solution to $t^2y'' - 2y = t^6$ with $t > 0$.

$y(t) = u(t)t^2 \Rightarrow y' = u't^2 + 2ut$, $y'' = u''t^2 + 2u't + 2u't + 2u$

$y'' = u''t^2 + 4u't + 2u$

$\Rightarrow t^2(u''t^2 + 4u't + 2u) - 2(ut^2) \overset{?}{=} t^6$

$\Rightarrow t^4u'' + 4t^3u' + \cancel{2t^2u} - \cancel{2t^2u} = t^6$

if you don't see this you can do integrating factor

$\underbrace{t^4u'' + 4t^3u'}_{} = t^6$

$\frac{d}{dt}(t^4u') = t^6$

$t^4u' = \frac{1}{7}t^7 + a_1$

$u'(t) = \frac{1}{7}t^3 + a_1t^{-4}$

$u(t) = \frac{1}{28}t^4 - \frac{a_1}{3}t^{-3} + a_2$

$$\boxed{\begin{aligned} y(t) &= u(t)t^2 = -\frac{a_1}{3}t^{-1} + a_2t^2 + \frac{1}{28}t^6 \\ y(t) &= c_1t^{-1} + c_2t^2 + \frac{1}{28}t^6 \end{aligned}}$$