Math 307 - Homework 4 - Dr. Loveless

Due WEDNESDAY, May 4th

Hand in your work in the order it is assigned (Staple all your work together before coming to class). This is a minimal list of problems, I strongly encourage you to do more problems than are assigned.

1. 3.3/7, 11, 16, 17, 19, 35, 36 (for 35, 36, see instructions)
2. 3.4/2, 6, 11, 14, 16, 23, 24, 25, 40 (for 40, see instructions)
3. 3.5/1, 15 (see instructions)

NOTES AND SPECIAL INSTRUCTIONS :

- 3.3/35, 36 and 3.4/40: First read problem 3.3/34 which explains how to transform the problem into an equivalent form that is constant coefficient. Do the transformation, solve for $y_1(x)$ and $y_2(x)$ as we have learned to do in 3.1, 3.3, and 3.4, then your final answer is $y_1(\ln(t))$, $y_2(\ln(t))$. For more information (for your own interest), read the supplemental comments on the back of this page concerning Euler equations.
- 3.5/1, 15: I will assign more 3.5 problems next week. My goal here is to give you an introductory opportunity to see the method of 3.5 in action.
  Instructions for 3.5/1:
  1. Find the general solution for $y'' - 2y' - 3y = 0$. (Use 3.1, 3.3, 3.4 methods)
  2. Write $Y(t) = Ae^{2t}$ and find $Y'(t)$ and $Y''(t)$.
     Substitute $Y(t)$ into $y'' - 2y' - 3y = 3e^{2t}$ and solve for A.
  3. The general solution is the sum of the solution from the first part and $Y(t)$.

  Instructions for 3.5/15:
  1. Find the general solution for $y'' + y' - 2y = 0$. (Use 3.1, 3.3, 3.4 methods)
  2. Write $Y(t) = At + B$ and find $Y'(t)$ and $Y''(t)$.
     Substitute $Y(t)$ into $y'' + y' - 2y = 2t$ and solve for A and B.
  3. The general solution is the sum of the solution from the first part and $Y(t)$.
  4. Use your initial conditions $y(0) = 0$ and $y'(0) = 1$ to solve for your two constants.

Additional comments about Euler Equations: Problem 3.3/34 in the book describes how to solve a special type of second order linear equation that is NOT constant coefficient. As you will see below this is only for a very special case and it requires a clever change of variable. In other words, I am showing you that we don't have nice general ways to solve 2nd order linear equations that aren't constant coefficient in this class. The best we really can do is to guess one solution and use reduction of order to find another. Concerning Euler equations, here is what problem 34 shows (You don't have to know how to derive this for the test, this is for your own interest):

Assume we are trying to solve $t^2\frac{d^2y}{dt^2} + \alpha t\frac{dy}{dt} + \beta y = 0$.

We make the change of variable $x = \ln(t)$. Meaning we are going to try to replace $\frac{d^2y}{dt^2}$ and $\frac{dy}{dt}$ by $\frac{d^2y}{dx^2}$ and $\frac{dy}{dx}$. Here is how that is done:

- First, since $x = \ln(t)$, we have $\frac{dx}{dt} = \frac{1}{t}$ and $\frac{d^2x}{dt^2} = -\frac{1}{t^2}$.
- Second, from calculus 1 and 3, we have $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$. Thus, $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$.
- Third, also from calculus 1 and 3, we have and $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy/dt}{dx/dt}\right) = \frac{(dx/dt)(d^2y/dt^2)-(dy/dt)(d^2x/dt^2)}{(dx/dt)^3}$. Multiplying both sides by $\frac{d^2x}{dt^2}$ and rearranging gives:

$$\frac{d^2y}{dx^2}\left(\frac{dx}{dt}\right)^2 = \frac{d^2y}{dt^2} - \frac{dy/dt}{dx/dt}\frac{d^2x}{dt^2} = \frac{d^2y}{dt^2} - \frac{dy}{dx}\frac{d^2x}{dt^2} \Rightarrow \frac{d^2y}{dt^2} = \frac{d^2y}{dx^2}\left(\frac{dx}{dt}\right)^2 + \frac{dy}{dx}\frac{d^2x}{dt^2}$$

- Last, replace $\frac{dx}{dt}$ and $\frac{d^2x}{dt^2}$ to get

$$\frac{dy}{dt} = \frac{1}{t}\frac{dy}{dx} \quad \text{and} \quad \frac{d^2y}{dt^2} = \frac{1}{t^2}\frac{d^2y}{dx^2} - \frac{1}{t^2}\frac{dy}{dx}.$$

Now substituting this into the differential equation and we get:

$$t^2\left(\frac{1}{t^2}\frac{d^2y}{dx^2} - \frac{1}{t^2}\frac{dy}{dx}\right) + \alpha t\left(\frac{1}{t}\frac{dy}{dx}\right) + \beta y = 0, \text{ which simplifies to } \frac{1}{t^2}\frac{d^2y}{dx^2} + (\alpha - 1)\left(\frac{1}{t}\frac{dy}{dx}\right) + \beta y = 0.$$

In other words, if you are given $t^2y'' + \alpha ty' + \beta y = 0$ (all derivatives in terms of t) and you make the change of variable $x = \ln(t)$, then you get the constant coefficient equation $y'' + (\alpha - 1)y' + \beta y = 0$ (all derivatives in terms of x).

Here is an example of what I want you to do in the homework on 3.3/35, 36, and 3.4/40:

Example: Solve $t^2y'' - 4ty' - 6y = 0$ (Note: $\alpha = -4$ and $\beta = -6$)

Solution: From above, the change of variable $x = \ln(t)$ gives an equation in terms of x of the form $y'' - 5y' - 6y = 0$.
The equation $r^2 - 5r - 6 = (r - 6)(r + 1) = 0$ has roots $r_1 = 6$ and $r_2 = -1$.
Thus, the solution (in terms of x) is $y(x) = c_1e^{6x} + c_2e^{-x}$.
Replacing $x = \ln(t)$ gives $y(t) = c_1e^{6\ln(t)} + c_2e^{-\ln(t)} = c_1t^6 + c_2t^{-1}$.