Chapter 3: Summary of Second Order Solving Methods

We only discussed solution methods for **linear** second order equations.

Constant Coefficient Methods: To solve an equation of the form: $ay'' + by' + cy = g(t)$.

Homogeneous (when $\mathbf{g(t) = 0}$): Solve $ar^2 + br + c = 0$ to get $r = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

$b^2 - 4ac > 0$	Two real roots: r_1 and r_2	General Solution: $y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$.
$b^2 - 4ac = 0$	Repeated root: r	General Solution: $y(t) = c_1 e^{rt} + c_2 t e^{rt}$.
$b^2 - 4ac < 0$	Complex roots: $r = \lambda \pm \omega i$	General Solution: $y(t) = c_1 e^{\lambda t}\cos(\omega t) + c_2 e^{\lambda t}\sin(\omega t)$.

Nonhomogeneous (when $\mathbf{g(t) \neq 0}$):

1. Solve the corresponding homogeneous equation and get independent solutions $y_1(t)$ and $y_2(t)$.
2. Find *any* particular solution, $Y(t)$, to $ay'' + by' + cy = g(t)$.
   - Option 1: If $g(t)$ is a product or sum of polynomials, exponentials, sines or cosines, then use **undetermined coefficients**.
   - Option 2: If $g(t)$ involves some function other than those mentioned above, then use **reduction of order** (or more generally, variation of parameters). See the discussion at the bottom of this page about reduction of order for a reminder of how this can be done.
3. General Solution: $y(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t)$.

Nonconstant Coefficient Methods: To solve an equation of the form: $y'' + p(t)y' + q(t)y = g(t)$.
Homogeneous (when $\mathbf{g(t) = 0}$):

1. Option 1: If the equation can be written as $P(x)y'' + Q(x)y' + R(x)y = 0$, then we say it is **exact** when $P''(x) - Q'(x) + R(x) = 0$. In 3.2/41-45, you see how to solve these.
   (a) Let $f(x) = Q(x) - P'(x)$.
   Note: $P(x)y'' + Q(x)y' + R(x) = 0$ is the same as $\frac{d}{dx}(P'(x)y') + \frac{d}{dx}(f(x)y) = 0$.
   (b) Integrate both sides to get $P'(x)y' + f(x)y = c_1$. Solve this 1st order equation (integrating factor!).
2. Option 2: Change the variable. The only examples we saw were **Euler equations** which take the form: $t^2y'' + \alpha t y' + \beta y = 0$. In 3.3/34-41, you see how to solve these.
   (a) Making the change of variable $x = \ln(t)$ leads to $y'' + (\alpha - 1)y' + \beta y = 0$.
   (b) Solve this constant coefficient equation (using methods above).
   (c) This gives a solution equation $y = y(x)$. Now replace x with $\ln(t)$.

Nonhomogeneous (when $\mathbf{g(t) \neq 0}$): To solve an equation of the form: $y'' + p(t)y' + q(t)y = g(t)$.

1. Solve the corresponding homogeneous equation and get a solution $y = y_1(t)$ (if possible, find a second independent solution as well $y_2(t)$).
2. Use **reduction of order**,
   (a) Write $y = u(t)y_1(t)$. And compute y' and y''
   (b) Plug y, y' and y'' into the original nonhomogeneous equation. Simplify to get a first order equation and solve for $u(t)$.
   (c) Then $y = u(t)y_1(t)$ will be the full general solution.
3. Or use variation of parameters from section 3.6 (you are not expected to know this for the exam).
4. General Solution: $y(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t)$