Chapter 2: Summary of First Order Solving Methods

Given $\frac{dy}{dt} = f(t,y)$ with $y(t_0) = y_0$.

1. LINEAR?
   If so, rewrite in the form $\dfrac{dy}{dt} + p(t)y = g(t)$. And use the integrating factor method!

2. SEPARABLE?
   If so, factor, separate and integrate: $\dfrac{dy}{dt} = f(t,y) = h(t)g(y) \Longrightarrow \displaystyle\int \frac{1}{g(y)}\, dy = \int h(t)\, dt.$

3. EXACT?:
   Rewrite in form $M(t,y) + N(t,y)\dfrac{dy}{dt} = 0$: if $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial t}$, then use the exact method.
   Integrate to get $\displaystyle\int M(t,y)\, dt + C_1(y)$ and $\displaystyle\int N(t,y)\, dy + C_2(t)$, combine to get $\Psi(t,y)$ and the solution is $\Psi(t,y) = C$.

4. SUBSTITUTION?:
   Let $u =$ 'some expression involving t and y', then differentiate with respect to t to get a relationship between $\dfrac{du}{dt}$ and $\dfrac{dy}{dt}$. Substitute to turn $\dfrac{dy}{dt} = f(t,y)$ into an equation involving only t and u. And HOPE! Hope that the new equation is one you can solve by one of our other methods. If I give you such a problem on the test, I will tell you the substitution to use.

Other Notes:

1. If you are asked to find an **explicit** solution, then your final answer needs to be in the form $y = y(x)$. In other words you must solve for y. If you do not (or cannot) solve for y in terms of t, then we say your answer is an **implicit** solution.

2. Remember to recognize any equilibrium solutions at the beginning. And you can also classify them as stable, unstable or semistable before you start (this also helps to check your work).

3. Remember to use your initial condition in the end!

4. If $f(t,y)$ is discontinuous or undefined at any t values, then that restricts the domain of our final answer. If $f(t,y)$ or $\dfrac{\partial f}{\partial t}(t,y)$ is discontinuous or undefined at any y, then that also restricts the domain/range of our solution. If the initial condition is at one of these discontinuities, then solutions may not exists and may not be unique.

5. You can always **check your final answer**! Here is how you check:
   (a) Take your solution and differentiate to find $\frac{dy}{dt}$. Substitute what you just found for $\frac{dy}{dt}$ in your differential equation. Also replace y by $y(t)$ in your differential equation. If both sides of the differential equation are equal, then you have a solution!
   (b) Also check your initial condition.
   (c) If your function works in the differential equation (makes both sides equal) and if your function satisfies the initial condition, then you will know with certainty that you have a solution!