2.7: Euler's Method

Given $\frac{dy}{dt} = f(t, y)$ with $y(t_0) = y_0$. The methods for finding explicit (or implicit) solutions are limited. We can solve only a small collection of special types of differential equations. In many applied problems numerical methods are essential. One of the most fundamental approximation methods is Euler's method which we describe here.

IDEA: Note that $\frac{dy}{dt}$ is the slope of the tangent line to $y(t)$. If we are given a point (t_0, y_0), we can directly evaluate $\frac{dy}{dt} = f(t_0, y_0)$ to get the slope of the tangent line at that point. In fact, we can get the equation for the tangent line $y = y_0 + f(t_0, y_0)(t - t_0)$ at that point. If we change t slightly (a small step of h, so that $t_1 = t_0 + h$) and compute the new y value from the tangent line, then we will get something very close to the actual value of the solution at that step. In terms of the formula, we get a new points $t_1 = t_0 + h$ and $y_1 = y_0 + f(t_0, y_0)h$. The idea is to repeat this process over and over again in order to find the path of the solution.

EULER'S METHOD: More formally, given $\frac{dy}{dt} = f(t, y)$ with $y(t_0) = y_0$ we approximate the path of the solution by:

1. STEP SIZE: First, we choose the step size, h, which is the size of the increments along the t-axis that we will use in approximation. Smaller increments tend to give more accurate answers, but then there are more steps to compute. We often use some value around $h = 0.1$ in our examples in this class (but in applications $h = 0.001$ is probably a better choice).

2. COMPUTE SLOPE: Compute the slope $\frac{dy}{dt} = f(t_0, y_0)$.

3. GET NEXT POINT: The next point is $t_1 = t_0 + h$ and $y_1 = y_0 + f(t_0, y_0)h$.

4. REPEAT: Repeat the last two steps with (t_1, y_1). Then repeat again with (t_2, y_2) and repeat again and again, until you get to the desired value of t.

It might help to make a table:

t	t_0	$t_1 = t_0 + h$	$t_2 = t_1 + h$	$t_3 = t_2 + h$	$\cdots$
y	y_0	$y_1 = y_0 + f(t_0, y_0)h$	$y_2 = y_1 + f(t_1, y_1)h$	$y_3 = y_2 + f(t_2, y_2)h$	$\cdots$

Here are two quick examples:

1. Let $\frac{dy}{dt} = 2t + y$, $y(1) = 5$. Using Euler's method with $h = 0.2$ approximate the value of $y(2)$.

t	1	1.2	1.4	1.6	1.8	2.0
y	5	6.4	8.16	10.352	13.0624	16.39488

   Here are the calculations for how I filled in the table above:

   (a) $f(1,5) = 2(1) + (5) = 7$, so $y(1.2) \approx 5 + 7(0.2) = 6.4$.
   (b) $f(1.2, 6.4) = 2(1.2) + (6.4) = 8.8$, so $y(1.4) \approx 6.4 + 8.8(0.2) = 8.16$.
   (c) $f(1.4, 8.16) = 2(1.4) + (8.16) = 10.96$, so $y(1.6) \approx 8.16 + 10.96(0.2) = 10.352$.
   (d) $f(1.6, 10.352) = 2(1.4) + (10.352) = 13.552$, so $y(1.8) \approx 10.352 + 13.552(0.2) = 13.0624$.
   (e) $f(1.8, 13.0624) = 2(1.4) + (13.0624) = 16.6624$, so $y(2) \approx 13.0624 + 16.6624(0.2) = 16.39488$.

   Aside: In this case, an explicit answer can be found $y(t) = 9e^{t-1} - 2(t+1)$. And note that the actual value is $y(2) = 9e - 6 \approx 18.4645$.
   So the answer we got is within 2 (which is a pretty big error).
   If you use $h = 0.1$, then it takes 10 steps to get to $y(2)$ and you get an approximation of $y(2) \approx 17.3437$.
   If you use $h = 0.01$, then it takes 100 steps to get to $y(2)$ and you get an approximation of $y(2) \approx 18.3433$.

2. Let $\frac{dy}{dt} = \frac{2}{ty} + \ln(y)$, $y(1) = 2$. Using Euler's method with $h = 0.5$ approximate the value of $y(3)$.

t	1	1.5	2	2.5	3
y	2	2.846574	3.603831	4.383571	5.213753

   Here are the calculations for how I filled in the table above:

   (a) $f(1,2) = \frac{2}{(1)(2)} + \ln(2) \approx 1.6931$, so
   $y(1.5) \approx 2 + 1.6931(0.5) \approx 2.846574$.
   (b) $f(1.5, 2.846574) = \frac{2}{(1.5)(2.846574)} + \ln(2.846574) \approx 1.514515$, so
   $y(2) \approx 2.846574 + 1.514515(0.5) \approx 3.603831$.
   (c) $f(2, 3.603831) = \frac{2}{(2)(3.603831)} + \ln(3.603831) \approx 1.55948$, so
   $y(2.5) \approx 3.603831 + 1.55948(0.5) \approx 4.383571$.
   (d) $f(2.5, 4.383571) = \frac{2}{(2.5)(4.383571)} + \ln(4.383571) \approx 1.660363$, so
   $y(3) \approx 4.383571 + 1.660363(0.5) \approx 5.213753$.

   Aside: There is no nice solution in terms of elementary functions. Using the basic numerical solver on Mathematica gives an approximation of $y(3) \approx 5.19232$ which is very close to our rough estimate.