2.6: Exact Equations

In this class, we'll define an **exact** equation as a first order differential equation of the form:

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \quad \text{where} \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

In such a case, a solution exists and can be found by reversing implicit differential (existence guaranteed assuming M, N, M_y, and N_x are continuous).

Here are some motivational examples. The three examples below all start with an equation that implicitly defines a function and asks you to find $\frac{dy}{dx}$ (you should remember doing such problems in Math 124, 125, and 126). The answer for $\frac{dy}{dx}$ is given in the first line, then some observation are made which should help motivate the exact equation method.

1. $x^2 + y^2 = 4$. Find $\frac{dy}{dx}$.

   Solution: Differentiating with respect to x gives: $2x + 2y\frac{dy}{dx} = 0$. Thus, $\frac{dy}{dx} = -x/y$.

   - $2x + 2y\frac{dy}{dx} = 0$ is an exact equation with $M(x,y) = 2x$ and $N(x,y) = 2y$. Observe $\frac{\partial M}{\partial y} = 0$ and $\frac{\partial N}{\partial x} = 0$ are the same.
   - $\displaystyle\int M(x,y)\,dx = \int 2x\,dx = x^2 + C_1(y)$ and $\displaystyle\int N(x,y)\,dy = \int 2y\,dy = y^2 + C_2(x)$

     Notice that the original equation is of the form $x^2 + y^2 = C$ (with $C = 4$).

2. $4x^3 + xy^4 = 7$. Find $\frac{dy}{dx}$.

   Solution: Differentiating with respect to x gives: $12x^2 + y^4 + 4xy^3\frac{dy}{dx} = 0$. Thus, $\frac{dy}{dx} = \frac{-12x^2 - y^4}{4xy^3}$.

   - $12x^2 + y^4 + 4xy^3\frac{dy}{dx} = 0$ is an exact equation with $M(x,y) = 12x^2 + y^4$ and $N(x,y) = 4xy^3$. Observe that $\frac{\partial M}{\partial y} = 4y^3$ and $\frac{\partial N}{\partial x} = 4y^3$ are the same.
   - $\displaystyle\int 12x^2 + y^4\,dx = 4x^3 + xy^4 + C_1(y)$ and $\displaystyle\int 4xy^3\,dy = xy^4 + C_2(x)$

     Notice that the original equation is of the form $4x^3 + xy^4 = C$ (with $C = 7$).

3. $3x^2e^y + 2y = 10$. Find $\frac{dy}{dx}$.

   Solution: Differentiating with respect to x gives: $6xe^y + 3x^2e^y\frac{dy}{dx} + 2\frac{dy}{dx} = 0$. Thus, $\frac{dy}{dx} = \frac{-6xe^y}{3x^2e^y+2}$.

   - $6xe^y + (3x^2e^y + 2)\frac{dy}{dx} = 0$ is an exact equation with $M(x,y) = 6xe^y$ and $N(x,y) = 3x^2e^y + 2$. Observe that $\frac{\partial M}{\partial y} = 6xe^y$ and $\frac{\partial N}{\partial x} = 6xe^y$ are the same.
   - $\displaystyle\int 6xe^y\,dx = 3x^2e^y + C_1(y)$ and $\displaystyle\int 3x^2e^y + 2\,dy = 3x^2e^y + 2y + C_2(x)$

     Notice that the original equation is of the form $3x^2e^y + 2y = C$ (with $C = 7$).

Summary of Observations and Notes:

- If $\psi(x,y) = C$ is the original, then $\psi(x,y)$ is a 'combined' expression that can be written in BOTH the forms $\displaystyle\int M(x,y)\,dx + C_1(y)$ and $\displaystyle\int N(x,y)\,dy + C_2(x)$.
- The proof of the fact that implicit differential always leads to the form $M(x,y) + N(x,y)\frac{dy}{dx} = 0$ with $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ is not part of this course (for more information take Math 126 and Math 324 and read about Clairaut's Theorem).

Exact Equations Method

1. FORM: Rewrite the equation in the form $M(x,y) + N(x,y)\frac{dy}{dx} = 0$.

2. CHECK CONDITION: Find $\dfrac{\partial M}{\partial y}$ and $\dfrac{\partial N}{\partial x}$.
   If they are different, give up! (This method won't work to solve the equation).
   If they are the same, keep going.

3. INTEGRATE: Evaluate $\displaystyle\int M(x,y)\,dx + C_1(y)$ and $\displaystyle\int N(x,y)\,dy + C_2(x)$.

4. FINAL ANSWER: The 'combined' function that looks like both expressions of previous part is $\psi(x,y)$ (write 'overlapping terms' only once). The final answer is $\psi(x,y) = c$ for some constant c. Use initial conditions to find c.

5. CHECK: Use implicit differentiation to check that your answer is correct!

ASIDE (Another perspective):
I've noticed that students often get confused by the presentation in textbooks for how to solve for $\psi(x,y)$, so I have stated the method above in terms of integrating and combining (like you saw in the examples on the previous page). Here is another (more common, textbook) way that you will see the INTEGRATE step described in textbooks.

To illustrate the textbook method, let's use the third example from the previous page which was
$6xe^y + (3x^2e^y + 2)\dfrac{dy}{dx} = 0$.

1. Integrate and write $\psi(x,y) = M(x,y)\,dx + h(y)$ where $h(y)$ is some function in terms of y. The integral will give all terms of $\psi(x,y)$ that involve the variable x.
   In the example this would look like $\psi(x,y) = \displaystyle\int 6xe^y\,dx = 3x^2e^y + h(y)$
   ($h(y)$ is the constant of integration, which might involve y since we are treating y like a constant).

2. Differentiate $\psi(x,y)$ with respect to y and equate the result to $N(x,y)$. Simplifying will give an equation of the form $h'(y) = ???$.
   Using the example, $\dfrac{\partial}{\partial y}\psi(x,y) = \dfrac{\partial}{\partial y}(3x^2e^y + h(y)) = 3x^2e^y + h'(y)$.
   We want this to equal $N(x,y)$, so we want $3x^2e^y + h'(y) = 3x^2e^y + 2$.
   In other words, we need $h'(y) = 2$.

3. Integrate with respect to y to get $h(y)$. And you are done (now you can write $\psi(x,y)$.
   Using the example, $h'(y) = 2$ means that $h(y) = 2y + C$. So the original function $\psi(x,y)$ must have the form $\psi(x,y) = 3x^2e^y + 2y$ and the final answer will be $3x^2e^y + 2y = C$ for some constant C. Notice this matches the answer on the previous page.

This is an efficient, formulaic way to cancel the 'overlap' (that is what you are doing in the middle step here), so that you don't have to integrate the same thing twice. If you take Math 324 (and some other upper level math classes), then you will use this technique again.
Use any method here that makes sense to you. Just make sure to check your work by implicitly differentiating your final answer!

Examples: (Leave your answers in implicit form)

1. Solve $\dfrac{dy}{dx} = -\dfrac{x}{y}$ with $y(0) = 4$.

   (a) FORM: $x + y\frac{dy}{dx} = 0$, so $M(x,y) = x$ and $N(x,y) = y$.

   (b) CHECK CONDITION: $\frac{\partial M}{\partial y} = 0$ and $\frac{\partial N}{\partial x} = 0$ are the same!

   (c) INTEGRATE: $\displaystyle\int x\,dx = \frac{1}{2}x^2 + C_1(y)$ and $\displaystyle\int y\,dy = \frac{1}{2}y^2 + C_2(x)$.

   (d) FINAL ANSWER: 'Combining' gives $\psi(x,y) = \frac{1}{2}x^2 + \frac{1}{2}y^2$.
   The general implicit answer is $\frac{1}{2}x^2 + \frac{1}{2}y^2 = C$.
   Using the initial condition of $y(0) = 4$ gives $0 + \frac{16}{2} = C$, so $C = 8$.
   Thus, $\frac{1}{2}x^2 + \frac{1}{2}y^2 = 8$ which is the same as $x^2 + y^2 = 16$.

2. Solve $(3y^2 + x^3)\dfrac{dy}{dx} = -2x - 3x^2y$ with $y(5) = 0$.

   (a) FORM: $(2x + 3x^2y) + (3y^2 + x^3)\frac{dy}{dx} = 0$, so $M(x,y) = 2x + 3x^2y$ and $N(x,y) = 3y^2 + x^3$.

   (b) CHECK CONDITION: $\frac{\partial M}{\partial y} = 3x^2$ and $\frac{\partial N}{\partial x} = 3x^2$ are the same!

   (c) INTEGRATE: $\displaystyle\int 2x + 3x^2y\,dx = x^2 + x^3y + C_1(y)$ and $\displaystyle\int 3y^2 + x^3\,dy = y^3 + x^3y + C_2(x)$.

   (d) FINAL ANSWER: 'Combining' gives $\psi(x,y) = x^2 + x^3y + y^3$.
   The general implicit answer is $x^2 + x^3y + y^3 = C$.
   Using the initial condition of $y(5) = 0$ gives $5^2 + 0 + 0 = C$, so $C = 25$.
   Thus, $x^2 + x^3y + y^3 = 25$.

3. Solve $\dfrac{dy}{dx} = \dfrac{y^2\sin(x) + 3x^2}{2y\cos(x)}$ with $y(0) = 3$.

   (a) FORM: $(-y^2\sin(x) - 3x^2) + (2y\cos(x))\frac{dy}{dx} = 0$, so $M(x,y) = -y^2\sin(x) - 3x^2$ and $N(x,y) = 2y\cos(x)$.

   (b) CHECK CONDITION: $\frac{\partial M}{\partial y} = -2y\sin(x)$ and $\frac{\partial N}{\partial x} = -2y\sin(x)$ are the same!

   (c) INTEGRATE: $\displaystyle\int -y^2\sin(x) - 3x^2\,dx = y^2\cos(x) - x^3 + C_1(y)$
   and $\displaystyle\int 2y\cos(x)\,dy = y^2\cos(x) + C_2(x)$.

   (d) FINAL ANSWER: 'Combining' gives $\psi(x,y) = y^2\cos(x) - x^3$.
   The general implicit answer is $y^2\cos(x) - x^3 = C$.
   Using the initial condition of $y(0) = 3$ gives $9 - 0 = C$, so $C = 9$.
   Thus, $y^2\cos(x) - x^3 = 9$.