2.4: First Order Theorems about Existence and Uniqueness

In this section you learn some basic theorems about when a solution will exist and be unique. These theorems are of practical and theoretical importance. Ultimately, these theorems give you a few 'trouble' spots to check before you start solving a differential equation. By 'trouble' spots, I mean initial values and locations where you can get no solution or multiple solutions. We certainly want to know that a solution exists and that the solution we find is the only solution!
Here are some rough, practical, oversimplified guides to how to approach first order problems in order to identify and avoid 'unusual' behavior.

NONLINEAR Initial Analysis Checklist
Given **any** first order equation of the form (linear or **nonlinear**): $\frac{dy}{dt} = f(t,y)$ with $y(t_0) = y_0$.

1. Identify any discontinuities of $f(t,y)$ or $\frac{\partial f}{\partial y}(t,y)$.
   At each discontinuity t-value draw a vertical line.
   At each discontinuity y-value draw a horizontal line.
   Solutions with the initial condition $y(t_0) = y_0$ are only guaranteed to exist and be unique around (t_0, y_0) and up to when the solution reaches the nearest horizontal or vertical 'discontinuity' line.

2. Note any equilibrium solutions.
   If you divide by a quantity when you separate, then make a note that the division is only valid if the quantity is NOT zero. This is vital for nonlinear systems. Sometimes, for nonlinear systems the equilibrium solution is NOT contained in the general solution you find from separating and integrating.

3. Then solve like you normally do. Separate and integrate.

LINEAR Initial Analysis Checklist
Given a **linear** first order equation of the form: $\frac{dy}{dt} + p(t)y = g(t)$ with $y(t_0) = y_0$.

1. Identify any discontinuities of $p(t)$ and $g(t)$.
   At each t-value corresponding to a discontinuity draw a vertical line.
   Solutions with the initial condition $y(t_0) = y_0$ are guaranteed to be valid around (t_0, y_0) and up to when the solution reaches the nearest vertical 'discontinuity' line.

2. It's good to note any equilibrium solutions, but it is not vital like in the nonlinear case.
   Because if you use the integrating factor method then you won't be dividing by y. And the equilibrium solution will be contained in the general solution!

3. Then solve like you normally do. Integrating factor!

NOTE: In this course we don't prove the facts from this section. Since $\frac{dy}{dt} = f(t,y)$, we might expect that at any discontinuity of $f(t,y)$ or $\frac{\partial f}{\partial y}(t,y)$ the slope field or the solution will either be undefined or change in some sort of dramatic way (which could lead to no solution or many solutions). For more explanations of the underpinnings of the theory, read 2.4 and 2.8 (and take higher level courses on differential equations).

Some Random Examples:

1. Consider $(t-2)\dfrac{dy}{dt}+y=\dfrac{t-2}{t}$ with $y(1)=4$.

    (a) This is a LINEAR equation in y. Divide by $(t-2)$ to get: $\dfrac{dy}{dt}+\dfrac{1}{t-2}y=\dfrac{1}{t}$.

    (b) Discontinuities? At $t=0$ and $t=2$. Since the initial condition is at $t=1$, the theorems from this section only guarantee a unique solution in the interval $0<t<2$.

    (c) Solution: $y(t)=\frac{t-2\ln(t)-5}{t-2}$ which is indeed valid and unique in the interval $0<t<2$.

2. Consider $\dfrac{dy}{dt}+\dfrac{1}{\cos^2(t)}y=e^{-\tan(t)}$ with $y(\pi)=2$.

    (a) This is a LINEAR equation in y. It is already in the correct form.

    (b) Discontinuities? $\frac{1}{\cos^2(x)}$ has a discontinuity at all values at which $\cos(t)=0$ (these are the same discontinuities that $\tan(t)$ has). The discontinuities are $t=\dots,-\frac{\pi}{2},\frac{\pi}{2},\frac{3\pi}{2},\dots$.
    The initial condition is at $t=\pi$, so the theorems from this section only guarantee a unique solution in the interval $\frac{\pi}{2}<t<\frac{3\pi}{2}$.

    (c) Solution: $y(t)=(t+2-\pi)e^{-\tan(t)}$ which is indeed valid and unique in the interval $\frac{\pi}{2}<t<\frac{3\pi}{2}$.

3. Consider $\dfrac{dy}{dt}=\dfrac{1}{y(t-3)^2}$ with $y(1)=-4$.

    (a) This is a NONLINEAR equation with $f(t,y)=\frac{1}{y(t-3)^2}$ and $\frac{\partial f}{\partial y}(t,y)=-\frac{1}{y^2(t-3)^2}$

    (b) Discontinuities? At $t=3$ and $y=0$. Since our initial condition is at $t_0=1$ and $y_0=-4$, the theorems from this section only guarantee a unique solution in the interval around t_0 for which the solution stays in the rectangular region $-\infty<t<3$ and $-\infty<y<0$.

    (c) Equilibrium? No equilibrium values.

    (d) General Solution: $y(t)=\pm\sqrt{\dfrac{-2}{t-3}+C}$.

    The initial condition $y(1)=-4$ gives $y(t)=-\sqrt{\dfrac{-2}{t-3}+15}$, which is only guaranteed to be unique when the function satisfies $-\infty<t<3$ and $-\infty<y<0$.

4. Consider $\dfrac{dy}{dt} = 2t(y-1)^{4/5}$ with $y(1) = 2$.

(a) This is a NONLINEAR equation with $f(t,y) = 2t(y-1)^{4/5}$ and $\frac{\partial f}{\partial y}(t,y) = \frac{8}{5}t(y-1)^{-1/5}$.

(b) Discontinuities? At $y = 1$. Since our initial condition is at $t_0 = 1$ and $y_0 = 2$, the theorems from this section only guarantee a unique solution in the interval around t_0 for which the solution stays in the rectangular region $-\infty < t < \infty$ and $1 < y < \infty$.

(c) Equilibrium? There is an equilibrium solution at $y = 1$.

(d) Separating and solving and you get 'general' answers of the form: $y(t) = 1 + \left(\dfrac{1}{5}t^2 + C\right)^5$.

The initial condition $y(1) = 2$ gives $y(t) = 1 + \left(\dfrac{1}{5}t^2 + \dfrac{4}{5}\right)^5$, which is only guaranteed to be unique when the function satisfies $-\infty < t < \infty$ and $1 < y < \infty$.

NOTE: If the initial condition is of the form $y(t_0) = 1$, then you are not guaranteed unique solutions because your initial condition $y_0 = 1$ is one of the points of discontinuity.
For example, consider $\dfrac{dy}{dt} = 2t(y-1)^{4/5}$ with $y(0) = 1$.
Using this initial condition in our 'general' answer gives $C = 0$ for a solution of $y(t) = 1 + \left(\frac{1}{5}t^2\right)^5 = 1 + \frac{1}{5^5}t^{10}$. This is indeed a solution (it satisfies the differential equation and this initial condition). However, there is another solution, the equilibrium solution $y(t) = 1$ which also satisfies the differential equation and the initial condition. So the solution is not unique!

5. Consider $\dfrac{dy}{dt} = 3y^3\sqrt{t}$ with $y(1) = \frac{1}{3}$.

(a) This is a NONLINEAR equation with $f(t,y) = 3y^3\sqrt{t}$ and $\frac{\partial f}{\partial t}(t,y) = 9y^2\sqrt{t}$.

(b) Discontinuities? We must have $t \geq 0$. Since our initial condition is at $t_0 = 1$ and $y_0 = 4$, the theorems from this section only guarantee a unique solution in the interval around t_0 for which the solution stays in the rectangular region $0 < t < \infty$ and $-\infty < y < \infty$.

(c) Equilibrium? There is an equilibrium solution at $y = 0$.

(d) Separating and solving and you get 'general' answers of the form: $y(t) = \dfrac{\pm 1}{\sqrt{C - 4t^{3/2}}}$.

The initial condition $y(1) = \frac{1}{3}$ gives $y(t) = \dfrac{1}{\sqrt{13 - 4t^{3/2}}}$, which is only guaranteed to be unique when the function satisfies $0 < t < \infty$ and $-\infty < y < \infty$.

NOTE: Don't forget the equilibrium solution when doing a problem like this. For example, consider $\dfrac{dy}{dt} = 3y^3\sqrt{t}$ with $y(1) = 0$. If you use this in our 'general' answer ($y(t) = \dfrac{\pm 1}{\sqrt{C - 4t^{3/2}}}$), you get $0 = \frac{\pm 1}{C-4}$ which has NO solution for C!!!
Our theorem guarantees that a solution exists and that it is unique, so where did our solution go? The answer is that the 'general' answer I gave did not contain all the answers. In a nonlinear problem, the equilibrium solutions may not be contained in the answer you get from separating and integrating. So the correct general answer for this differential equation is:
$y(t) = 0$ OR $y(t) = \dfrac{\pm 1}{\sqrt{-t^{3/2} + C}}$.
For the initial condition $y(1) = 0$, the unique solution is the equilibrium solution $y(t) = 0$.