2.3 Mathematical Modeling Examples for Lecture

1. **A mixing problem**:

A 1000 liter (L) tank initially contains 400L of water, in which there is 50 kilograms (kg) of dissolved salt.
Fluid that contains 0.3 kg/L of salt (*i.e.* it is 30% salt) is coming into the tank at 30 L/min.
The tank has a drain spout at the bottom and fluid is coming out at 10 L/min.
How much salt is in the tank when it starts to overflow?

STEP 1 Label:	y(t) = amount of salt in kg	
STEP 2 Find volume of water:	V(t) =??? (starts at 400L adds 30 – 10 = 20L every minute)	
STEP 3 Differential Equation:	dy/dt = RATE IN – RATE OUT	(should be in kg/min)
	= ??? - ???	
	y(0) = ???	(should be in kg)
STEP 4 Solve		

2. **Temperature**:

(a) A cup contains hot tea that is initially 200 degrees Fahrenheit (^{o}F).
The room temperature is 70^{o}F. Assume the rate of change of the temperature of the tea is proportional to the difference in temperature between tea and its surroundings. (Newton's Law of cooling).
Assume the cooling (proportionality) constant for the cup is known to be -0.1.
What is the differential equation?
(b) What if instead, the cup of tea is left outside and the temperature outside is given by the formula

$$T_s(t) = 70 + 20\sin\left(\frac{2\pi}{24}t\right)$$

What is the differential equation?

STEP 1 Label:	T(t) = temperature of tea in ^{o}F
	T_s = temperature of surroundings
STEP 2 Differential Equation:	dT/dt = k (T – T_s)
	T(0) = ???
STEP 3 Solve	

3. **Savings and Loans:**

(a) You deposit $10,000 into an account that earns 3% annually, compounded continuously. What is the value in 5 years?
(b) You deposit $10,000 into an account that earns 3% annually, compounded continuously. And you deposit an additional $2,000 per year in the same account. What is the value in 5 years?
(c) You are paying back $30,000 in student loans. The interest rate is 5% annually, compounded continuously. How much should you pay each year in order to pay off the loan in 10 years?

STEP 1 Label:	A(t) = balance of account after t years in dollars
STEP 2 Differential Equation:	dA/dt = rA ± K
	r = annual rate, rA = interest earned in a year
	± K = additional amount deposited/withdrawn each year
	A(0) = ???
STEP 3 Solve	

4. **Air Resistance**:

(a) You drop a ball from 10 meters in the air. The only force on the ball is force due to gravity (you ignore air resistance). What is the differential equation?

(b) You drop a ball from 10 meters in the air. Assume the force due to air resistance is proportional to the velocity of the ball. What is the differential equation? Solve the differential equation!

(c) You drop a ball from 10 meters in the air. Assume the force due to air resistance is proportional to the square of the velocity of the ball. What is the differential equation?

(d) A spherical object of mass m = 1 kg is shot straight upward into the air from the ground with initial velocity 250 m/sec (≈559 mph). Assume the force due to air resistance is proportional to the velocity with proportionality constant k = 0.2 kg/sec (this constant is reasonable for a sphere on earth in air with these units). What is the maximum height reached by the object?

STEP 1 Label: You get to make a choice

Option 1: h(t) = height of object from the ground (up is pos, down is neg)
In option 1, force due to gravity is negative and
force due to air resistance is positive when object is falling and
negative when object is rising.

Option 2: x(t) = distance object has fallen (down is pos, up is neg)
In this option, force due to gravity is positive and
force due to air resistance is negative when object is falling and
positive when object is rising.

STEP 2 Differential Equation: Since ma = F, we have m dv/dt = sum of forces.

For option 1: (a) $m\,dv/dt = -mg$ (no air resistance)
(b) $m\,dv/dt = -mg - kv$ (air resistance proportional to velocity)
(c) if air resistance is proportional to the square of the velocity, then you have to split up the problem;
when the object is moving upward: $m\,dv\,dt = -mg - kv^2$
when the object is moving downward: $m\,dv\,dt = -mg + kv^2$

For option 2: (a) $m\,dv/dt = mg$ (no air resistance)
(b) $m\,dv/dt = mg - kv$ (air resistance proportional to velocity)
(c) if air resistance is proportional to the square of the velocity, then you have to split up the problem;
when the object is moving upward: $m\,dv\,dt = mg + kv^2$
when the object is moving downward: $m\,dv\,dt = mg - kv^2$

STEP 3 Solve