Math 307A, Winter 2014 **Final Exam** **University of Washington**

Your Name

Student ID #

- Cellphones off please!
- Please box all of your answers.
- You are allowed one two-sided handwritten notesheet for this midterm. You may use a scientific calculator; graphing calculators and all other course-related materials may not be used.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct.
- Give your answers in exact form (for example $\frac{\pi}{3}$ or $e^{-5\sqrt{3}}$) unless explicity stated otherwise by the question.
- If you need more room, use the backs of the pages and indicate that you have done so.
- Raise your hand if you have a question.
- There is a table of Laplace transforms and rules at the back of this exam. You may quote and use any of the formulas and rules in the table as is without having to derive them from scratch.
- This exam has 10 pages, plus this cover sheet. Please make sure that your exam is complete.
- You have 110 minutes to complete the exam.

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
Total	80	

1. (10 points) Find the explicit solution to the initial value problem

$$y' + t^2 y - t^2 = 0, \qquad y(0) = -1$$

Your answer should be in the form $y = g(t)$, where $g(t)$ contains no undetermined constants.

2. (10 points) Use the method of undetermined coefficients to find the **particular solution** to the following differential equation:

$$y'' + 9y = te^{-t} - 1.$$

Your answer should be a function $Y(t)$ with no undetermined constants in it.

3. (10 total points) Consider following initial value problem:

$$\frac{dy}{dx} = \cos^2(x)\cos^2(y), \qquad y(0) = y_0,$$

where y_0 is a given constant.

(a) (2 points) By using the existence and uniqueness theorem for first-order differential equations, find the values of y_0, if any, for which the IVP is *not* guaranteed a unique solution in some time interval about $x = 0$.

(b) (6 points) Solve the IVP for the initial value $y_0 = 0$. [Hint: $\frac{d}{dy}\tan(y) = \sec^2(y)$]

(c) (2 points) Let $y = \phi(x)$ be the solution you found in part (b). What is the limiting value of the solution i.e. what is $\lim_{x\to\infty}\phi(x)$? Be sure to justify your answer.

4. (10 points) In each part of this question you are given a function $y(t)$ which is the general solution to a constant-coefficient homogeneous 2nd-order differential equation. Write down the differential equation that that function satisfies. Your answer should be a DE in the form $ay'' + by' + cy = 0$ for some values of a, b and c.

   Each part is worth 2 points. You don't need to show your working to get full credit for this question.

   (a) $y(t) = c_1 \cos(6t) + c_2 \sin(6t)$

   (b) $y(t) = c_1 e^{2t} + c_2 e^{-3t}$

   (c) $y(t) = c_1 + c_2 t$

   (d) $y(t) = c_1 e^{-t} + c_2 t e^{-t}$

   (e) $y(t) = c_1 e^{4t} \cos(2t) + c_2 e^{4t} \sin(2t)$

5. (10 points) Compute the inverse Laplace transform of the following function. Your answer should be a function $f(t)$. You may quote any formula or rule given in the Laplace transform formula sheet at the back of the exam paper.

$$F(s) = \frac{s^2 + 2s - 2}{s^3 - s}$$

6. (10 total points) A series circuit contains a capacitor of 4×10^{-4} F and an inductor of 1 H. The charge on the capacitor and the current in the circuit are both initially zero. Starting at time $t = 0$ an external voltage of $300\cos(40t)$ volts is applied to the circuit, where t is measured in seconds. Resistance is negligible. Consider the differential equation governing the charge $Q(t)$ in Coulombs on the capacitor as a function of time.

   To answer the following questions you may use known formulae to save time, but if so be sure to state the formula as you've seen it in class.

   (a) (3 points) Write down an initial value problem describing the charge on the capacitor as a function of time.

   (b) (2 points) What is the natural frequency of this system?

   (c) (2 points) The solution to the IVP above will exhibit beats. Write down the beat (angular) frequency of the solution in radians/sec.

   (d) (3 points) What is the maximum amount of charge that the capacitor will hold?

7. (10 total points) A heavy block of mass 1 kg is placed on a flat surface and attached to a horizontal spring. When the block is displaced 25 cm to the right of its equilibrium position the spring exerts a restoring force of 1 Newton to the left. Friction acts on the block proportional to its velocity such that when its speed is 1 m/s the block experiences a drag force of 4 Newtons.

   At time $t = 0$ seconds the block is at its equilibrium position traveling with a velocity of 1 m/s (i.e. traveling to the right). At $t = 1$ seconds a motor is switched on which exerts a force of $t - 1$ Newtons on the block. At $t = 4$ seconds the motor is switched off, and no external force acts on the block from thereon.

   (a) (2 points) Rewrite the forcing function $g(t)$ using Heaviside functions $u_c(t)$. Your answer should be expressible as a linear combination of $u_c(t)$'s each multiplied by some function of t.

   (b) (3 points) Establish an initial value problem that models the position of the block for $t \geq 0$.

   (c) (5 points) Let $y = \phi(t)$ be the solution to the IVP above. Compute the Laplace transform $\Phi(s)$ of the solution as a function of s.

   [NB: you do not need to fully solve the IVP to answer this part of the question.]

8. (10 total points + 4 bonus points) A two-way pump attached attached to a reservoir pumps water into and then out of the reservoir at a rate of $2000\sin(t)$ liters per hour, where t is measured in hours. At time $t = 0$ a valve at the bottom of the reservoir is opened and it begins to drain at a rate proportional to the amount of water in the reservoir. The reservoir initially contains 10000 liters of water, and the initial outflow rate is measured to be 1000 liters per hour.

   (a) (3 points) Establish an initial value problem that models the volume of water in the reservoir at time t.

   (b) (7 points) Solve the initial value problem to find the number of liters of water in the reservoir at time t.

[Question continued on the next page]

(c) (4 bonus points) Estimate the point in time when the reservoir first runs dry.

Table of Laplace Transforms

In this table, n always represents a positive integer, and a and c are real constants.

$f(t) = \mathscr{L}^{-1}[F(s)]$	$F(s) = \mathscr{L}(f(t))$	
1	$\frac{1}{s}$	$s > 0$
e^{at}	$\frac{1}{s-a}$	$s > a$
t^n, n a positive integer	$\frac{n!}{s^{n+1}}$	$s > 0$
$t^n e^{ct}$, n a positive integer	$\frac{n!}{(s-c)^{n+1}}$	$s > c$
t^a, $a > -1$	$\frac{\Gamma(a+1)}{s^{a+1}}$	$s > 0$
$\cos(at)$	$\frac{s}{s^2+a^2}$	$s > 0$
$\sin(at)$	$\frac{a}{s^2+a^2}$	$s > 0$
$\cosh(at)$	$\frac{s}{s^2-a^2}$	$s > \lvert a\rvert$
$\sinh(at)$	$\frac{a}{s^2-a^2}$	$s > \lvert a\rvert$
$e^{ct}\cos(at)$	$\frac{s-c}{(s-c)^2+a^2}$	$s > c$
$e^{ct}\sin(at)$	$\frac{a}{(s-c)^2+a^2}$	$s > c$
$u_c(t)$	$\frac{e^{-cs}}{s}$	$s > 0$
$u_c(t)f(t-c)$	$e^{-cs}F(s)$	
$e^{ct}f(t)$	$F(s-c)$	
$f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right)$	$c > 0$
$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \ldots - f^{(n-1)}(0)$	
$t^n f(t)$	$(-1)^n F^{(n)}(s)$	