1. (5 points) Solve the initial value problem

$$y' + 2y = te^{-2t}, \qquad y(1) = 0.$$

Solution: The integrating factor is $\mu(t) = e^{2t}$. So we have

$$\begin{aligned}
\frac{d}{dt}\left(e^{2t}y\right) &= te^{-2t}e^{2t} \\
e^{2t}y &= \int t\,dt \\
e^{2t}y &= \frac{1}{2}t^2 + c \\
y &= \left(\frac{1}{2}t^2 + c\right)e^{-2t}
\end{aligned}$$

Using the initial conditions, we find $c = -\frac{1}{2}$, and this gives the final answer.

2. (5 points) Let $y(t)$ be a solution to the differential equation

$$y' = (y-1)(y-2)^2, \qquad y(1) = y_0.$$

Determine all possible values of y_0 so that $\lim_{t\to\infty} y(t) = 2$.

Solution: The differential equation has constant solutions at $y = 1$ and $y = 2$. If $y < 1$, then $y' < 0$ so y is decreasing. If $y > 1$, then $y' > 0$ so y is increasing. The limit will be 2 if $1 < y_0 \leq 2$.

3. (5 points) A tank contains 2 kg of salt disolved in 500 L of water. Fresh water flows in at a rate of r L/s, and mixed solution flows out at the same rate.

   Determine the value of r so that the amount of salt in the tank is reduced by half in exactly 1 hour.

Solution: The differential equation is $Q' = -\frac{Qr}{500}$, with $Q(0) = 2$. This is both separable and linear. Either way you solve it, you get $Q = 2e^{-rt/500}$. Using $Q(3600) = 1$, you find $\frac{1}{2} = e^{-3600r/500}$, which means $r = \frac{5}{36} \ln 2$.

4. (5 points) Suppose you have an object of mass $1\,\mathrm{kg}$ hanging from a spring with spring coefficient $8\,\mathrm{N/m}$ and damping coefficient $2\,\mathrm{N/(m/s)}$. An external force (measured in Newtons) of $F_0 \cos 2t$ is applied to the system.

   Determine the amplitude of the steady state response.

   Solution: Using the method of undetermined coefficients, use $Y = A\cos 2t + B\sin 2t$. This gives equations $4A + 4B = F_0$ and $4A - 4B = 0$, which means $A = B = F_0/8$. So the amplitude is $\sqrt{A^2 + B^2} = F_0\sqrt{2}/8$.

5. (5 points) Solve the initial value problem

$$y'' + 7y' + 6y = \begin{cases} 0, & 0 \le t < 2 \\ e^{-2t}, & 2 \le t \end{cases}, \quad y(0) = 3, \quad y'(0) = 7.$$

Solution: The right hand side can be written $e^{-2t}u_2(t)$, and the Laplace transform is $e^{-4}e^{-2s}\mathcal{L}\left\{\frac{1}{s+2}\right\}$.

After taking Laplace transforms and solving for Y, we get

$$\begin{aligned} Y &= \frac{1}{(s+2)(s+1)(s+6)}e^{-2s}e^{-4} + \frac{3s+28}{(s+1)(s+6)} \\ &= \left(\frac{1/5}{s+2} - \frac{1/4}{s+6} + \frac{1/20}{s+1}\right)e^{-2s}e^{-4} + \frac{5}{s+1} - \frac{2}{s+6} \end{aligned}$$

The inverse Laplace transform is

$$y(t) = u_2(t)e^{-4}\left(\frac{1}{5}e^{-2(t-2)} - \frac{1}{4}e^{-6(t-2)} + \frac{1}{20}e^{-(t-2)}\right) + 5e^{-t} - 2e^{-6t}.$$

6. (5 points) Consider the initial value problem

$$y'' + 9y = A\delta_c(t), \quad y(0) = 0, \quad y'(0) = 2,$$

where $\delta_c(t)$ is an impulse function (also written $\delta(t-c)$), and A and c are POSITIVE constants.

Find values of A and c so that $y(t) = 0$ for all $t > c$.

Solution: Taking the Laplace transform of both sides and solving for Y, we get

$$Y = Ae^{-cs}\left(\frac{1}{s^2+9}\right) + \frac{2}{s^2+9}$$

Taking the inverse Laplace transform, we get

$$y(t) = \frac{2}{3}\sin 3t + \frac{A}{3}\sin(3t-3c)u_c(t)$$

The only way to get zero is if $A = 2$ and $3c = n\pi$ for n odd. For example, c could be $\pi/3$ or π.

Answer: $A =$ _________ $c =$ _________