1. (13 pts)

(a) Find the general explicit solution to $ty' - 2y = t^6$.

$$y' - \frac{2}{t}y = t^5 \Rightarrow \mu(t) = e^{\int -\frac{2}{t}dt} = e^{-2\ln(t)} = e^{\ln(t^{-2})} = t^{-2}$$

$$t^{-2}y' - 2t^{-3}y = t^3$$

$$\frac{d}{dt}(t^{-2}y) = t^3$$

$$t^{-2}y = \frac{1}{4}t^4 + C$$

$$\boxed{y = \frac{1}{4}t^6 + Ct^2}$$

(b) Find the explicit solution to $y' = 4xy^2e^{2x}$ with $y(0) = 4$.

$$\int \frac{1}{y^2}\,dy = \int 4xe^{2x}\,dx$$

$u = 4x$ $\quad dv = e^{2x}dx$

$du = 4dx$ $\quad v = \frac{1}{2}e^{2x}$

$$-\frac{1}{y} = 2xe^{2x} - \int 2e^{2x}dx$$

$$-\frac{1}{y} = 2xe^{2x} - e^{2x} + C$$

$$y = \frac{-1}{(2xe^{2x} - e^{2x} + C)}$$

$$y(0) = 4 \Rightarrow 4 = \frac{-1}{-1 + C} \Rightarrow -4 + 4C = -1$$

$$4C = 3$$

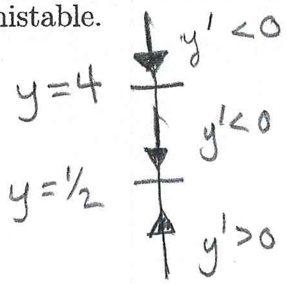
$$C = 3/4$$

$$\boxed{y = \frac{-1}{2xe^{2x} - e^{2x} + 3/4}}$$

2. (14 pts)

(a) Consider $y' = (1-2y)(y-4)^2$.

i. Determine the critical (equilibrium) points and classify each one as stable, unstable or semistable.



$$\boxed{\begin{array}{l} y(t) = 4, \text{ semistable} \\ y(t) = \frac{1}{2}, \text{ stable} \end{array}}$$

ii. Let $y(t)$ be the solution that satisfies the given differential equation with the initial condition $y(1) = 3$.
Use Euler's method with $h = 0.1$ to approximate the value of $y(1.1)$.

$$y' = (1-2(3))((3)-4)^2 = (1-6)(1) = -5$$

$$y(1.1) \approx 3 + (-5)(0.1) = \boxed{2.5}$$

(b) A baseball is dropped from an airplane. The mass of a baseball is about 0.2 kg. The force due to air resistance is proportional, and in opposite direction, to velocity with proportionality constant k (where $k > 0$).
Just like we did in homework, assume there are two forces acting on the ball: the force due to gravity and the force due to air resistance. (Recall: Newton's second law says $ma = F$ and the acceleration due to gravity is 9.8 meters/second^2.)

i. Give the differential equation and initial conditions for the velocity $v(t)$. (Do not solve)

$$\boxed{m \frac{dv}{dt} = -mg - kv} \qquad \boxed{v(0) = 0} \leftarrow \text{"DROPPED"}$$

$$m = 0.2 \qquad g = 9.8$$

ii. The value of $\lim_{t \to \infty} v(t)$ is called the terminal velocity. For a baseball, terminal velocity is known to be about 42 meters/second. Using this fact, find the value of the proportionality constant k. (Hint: You do NOT need to solve the differential equation).

$$0.2 \frac{dv}{dt} = \underbrace{-(0.2)(9.8)}_{1.96} - kv \stackrel{?}{=} 0$$

$$v = \frac{1.96}{-k} = \text{terminal velocity}$$

$$\frac{1.96}{-k} = -42$$

$$\Rightarrow \boxed{k = \frac{1.96}{42} \approx 0.04\overline{6} \ \frac{N}{m/s}}$$

3. (10 pts) Some cookie dough with an initial temperature of 40 degrees Fahrenheit is placed in an oven and the oven is turned on. The temperature of the oven is given by $f(t) = 350 - 280e^{-t/2}$ degrees Fahrenheit where t is in minutes. Assume the differential equation for the temperature of the cookie dough, $y(t)$, is given by

$$\frac{dy}{dt} = -\frac{1}{2}\left(y - 350 + 280e^{-t/2}\right).$$

Solve the differential equation to find the temperature of the cookie dough, $y(t)$, at time t minutes. (Hint: It's linear!)

$$\frac{dy}{dt} + \frac{1}{2}y = 175 - 140e^{-\frac{1}{2}t}$$

$$\mu(t) = e^{\frac{1}{2}t}$$

$$e^{\frac{1}{2}t}\frac{dy}{dt} + \frac{1}{2}e^{\frac{1}{2}t}y = 175e^{\frac{1}{2}t} - 140$$

$$\frac{d}{dt}\left(e^{\frac{1}{2}t}y\right) = 175e^{\frac{1}{2}t} - 140$$

$$e^{\frac{1}{2}t}y = 350e^{\frac{1}{2}t} - 140t + C$$

$$\boxed{y(t) = 350 - 140te^{-\frac{1}{2}t} + Ce^{-\frac{1}{2}t}}$$

$$y(0) = 40 \Rightarrow 40 = 350 - 0 + C$$

$$C = -310$$

$$\boxed{y(t) = 350 - 140te^{-\frac{1}{2}t} - 310e^{-\frac{1}{2}t}}$$

4. (12 pts) A certain mass-spring system satisfies $mu'' + 3u' + u = 0$, where m is the mass of the object attached to the end of the spring. The initial conditions are $u(0) = 3$ and $u'(0) = 0$.

(a) For what masses, m, will the system exhibit (damped) oscillations?
(Your answer will be a range of values)

$$\gamma < 2\sqrt{mk}$$

$$3 < 2\sqrt{m}$$

$$\frac{3}{2} < \sqrt{m}$$

$$\boxed{\frac{9}{4} < m} \text{ kg}$$

(b) Find the quasi-period of the solution if $m = 5$.

$$5r^2 + 3r + 1 = 0$$

$$r = \frac{-3 \pm \sqrt{9 - 20}}{10}$$

$$r = -\frac{3}{10} \pm \frac{i\sqrt{11}}{10} \qquad \mu = \frac{\sqrt{11}}{10}$$

$$\frac{2\pi}{\sqrt{11}/10} = \boxed{\frac{20\pi}{\sqrt{11}}} \text{ seconds}$$

(c) Find the solution $u(t)$ if $m = 2$. (Use the initial conditions.)

$$2r^2 + 3r + 1 = 0$$

$$(2r+1)(r+1) = 0$$

$$r = -1, \quad r = -\tfrac{1}{2}$$

$$u(t) = c_1 e^{-t} + c_2 e^{-\frac{1}{2}t} \qquad u(0) = 3 = c_1 + c_2$$

$$u'(t) = -c_1 e^{-t} - \tfrac{1}{2} c_2 e^{-\frac{1}{2}t} \qquad u'(0) = 0 = -c_1 - \tfrac{1}{2} c_2$$

$$3 = \tfrac{1}{2} c_2 \qquad c_2 = 6$$

$$c_1 = -3$$

$$\boxed{u(t) = -3e^{-t} + 6e^{-\frac{1}{2}t}}$$

5. (13 pts) Give the solution to $y'' + 4y = te^{2t}$ with $y(0) = 0$ and $y'(0) = 0$.

$r^2+4=0 \Rightarrow r = \pm 2i \Rightarrow c_1\cos(2t)+c_2\sin(2t)$

$Y(t) = (At+B)e^{2t}$

$Y'(t) = Ae^{2t} + 2(At+B)e^{2t} = (3At+2B)e^{2t}$

$Y''(t) = 2Ae^{2t} + 2Ae^{2t} + 4(At+B)e^{2t} = (4At+4A+4B)e^{2t}$

$y''+4y \stackrel{?}{=} te^{2t}$

$(4At+4A+4B)e^{2t} + 4(At+B)e^{2t} \stackrel{?}{=} te^{2t}$

$(8At+4A+8B)e^{2t} \stackrel{?}{=} te^{2t}$

$8A = 1 \Rightarrow A = 1/8$

& $4A+8B=0 \Rightarrow \frac{1}{2}+8B=0 \Rightarrow 8B=-1/2 \Rightarrow B=-1/16$

$y(t) = c_1\cos(2t)+c_2\sin(2t) + \left(\frac{1}{8}t - \frac{1}{16}\right)e^{2t}$

$y(0)=0 \Rightarrow c_1 - \frac{1}{16} = 0 \Rightarrow c_1 = 1/16$

$y'(t) = -2c_1\sin(2t) + 2c_2\cos(2t) + \frac{1}{8}e^{2t} + 2\left(\frac{1}{8}t-\frac{1}{16}\right)e^{2t}$

$y'(0)=0 \Rightarrow 2c_2 + \frac{1}{8} - 1/8 = 0 \qquad c_2 = 0$

$$\boxed{y(t) = \frac{1}{16}\cos(2t) + \left(\frac{1}{8}t - \frac{1}{16}\right)e^{2t}}$$

6. (14 pts)

(a) Give the form of a particular solution to $y'' - 4y' + 4y = 5 + te^{2t}$. (Do not solve, just give the form you would use for undetermined coefficients. Your answer will involved 'A, B, ...').

$r^2 - 4r + 4 = 0$

$(r-2)^2 = 0$

$r = 2$

$c_1 e^{2t} + c_2 t e^{2t}$

$A + \underbrace{(Bt + C)e^{2t}}$

TYPICAL FORM

BUT e^{2t} & te^{2t}

ARE BOTH HOMOGENEOUS SOL'NS

$$\boxed{A + t^2(Bt + C)e^{2t} = A + (Bt^3 + Ct^2)e^{2t}}$$

(b) Use the Laplace transform table (and step functions) to answer these questions:

i. Find the Laplace transform, $\mathcal{L}\{f(t)\}$, for the function $f(t) = \begin{cases} 3 & , 0 \le t < 6; \\ t + \cos(t-6) & , t \ge 6. \end{cases}$

$f(t) = 3 + (t - 3 + \cos(t-6))u_6(t)$

$$\mathcal{L}\{f(t)\} = \frac{3}{s} + \mathcal{L}\{((t-3) + \cos(t-6))u_6(t)\}$$

$$= \frac{3}{s} + e^{-6s}\mathcal{L}\{(t+6-3) + \cos(t+6-6)\}$$

$$= \boxed{\frac{3}{s} + e^{-6s}\left(\frac{3}{s} + \frac{1}{s^2} + \frac{s}{s^2+1}\right)}$$

ii. Find the inverse Laplace transform $\mathcal{L}^{-1}\left\{e^{-2s}\frac{2}{s^3} - e^{-5s}\frac{4}{s-8}\right\}$.

$$u_2(t)\mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\}(t-2) - u_5(t)\mathcal{L}^{-1}\left\{\frac{4}{s-8}\right\}(t-5)$$

$$\boxed{u_2(t)(t-2)^2 - 4u_5(t)e^{8(t-5)}}$$

7. (12 pts) Use the Laplace transform table (and algebra) to answer these questions:

(a) Find the Laplace transform of both sides of $y' = 3te^{4t} + 2t^3$ with $y(0) = 5$ and solve for $\mathcal{L}\{y\}$. (Don't do partial fraction and don't solve for $y(t)$, just stop when you get $\mathcal{L}\{y\}$ by itself and the other side all in terms of s.)

$$\mathcal{L}\{y'\} = 3\mathcal{L}\{te^{4t}\} + 2\mathcal{L}\{t^3\}$$

$$s\mathcal{L}\{y\} - \underbrace{y(0)}_{5} = \frac{3}{(s-4)^2} + 2\frac{3!}{s^4}$$

$$s\mathcal{L}\{y\} = 5 + \frac{3}{(s-4)^2} + \frac{12}{s^4}$$

$$\boxed{\mathcal{L}\{y\} = \frac{5}{s} + \frac{3}{s(s-4)^2} + \frac{12}{s^5}}$$

(b) Find the inverse Laplace transform, $\mathcal{L}^{-1}\left\{\frac{1}{s^2(s-2)} + \frac{1}{s^2+6s+13}\right\}$.

$$\frac{1}{s^2(s-2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-2} \Rightarrow 1 = As(s-2) + B(s-2) + Cs^2$$

$s=0 \Rightarrow B = -\frac{1}{2}$

$s=2 \Rightarrow C = 1/4$

coeff. of $s^2 \Rightarrow A + C = 0 \Rightarrow A = -1/4$

$$s^2+6s+13 = s^2+6s+9-9+13 = (s+3)^2+4$$

$$\mathcal{L}^{-1}\left\{\frac{-1/4}{s} - \frac{1/2}{s^2} + \frac{1/4}{s-2} + \frac{1}{(s+3)^2+4}\right\}$$

$$\boxed{-\frac{1}{4} - \frac{1}{2}t + \frac{1}{4}e^{2t} + \frac{1}{2}e^{-3t}\sin(2t)}$$

8. (12 pts) Solve $y'' + y = \begin{cases} 1 & , 0 \le t < 3; \\ 5 & , t \ge 3. \end{cases}$ with initial conditions $y(0) = 0$, $y'(0) = 2$.

$$y'' + y = 1 + 4u_3(t)$$

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{1\} + 4\mathcal{L}\{u_3(t)\}$$

$$s^2\mathcal{L}\{y\} - s\underbrace{y(0)}_{0} - \underbrace{y'(0)}_{2} + \mathcal{L}\{y\} = \frac{1}{s} + 4\frac{e^{-3s}}{s}$$

$$(s^2+1)\mathcal{L}\{y\} = 2 + \frac{1}{s} + 4\frac{e^{-3s}}{s}$$

$$\mathcal{L}\{y\} = \frac{2}{s^2+1} + \frac{1}{s(s^2+1)} + 4e^{-3s}\frac{1}{s(s^2+1)}$$

$$\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1} \Rightarrow 1 = A(s^2+1) + (Bs+c)s$$

$$1 = (A+B)s^2 + Cs + A$$

$$A = 1, \quad C = 0, \quad A + B = 0 \Rightarrow B = -1$$

$$\mathcal{L}\{y\} = \frac{2}{s^2+1} + \frac{1}{s} - \frac{s}{s^2+1} + 4e^{-3s}\left(\frac{1}{s} - \frac{s}{s^2+1}\right)$$

$$y(t) = 2\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + 4\mathcal{L}^{-1}\left\{e^{-3s}\left(\frac{1}{s} - \frac{s}{s^2+1}\right)\right\}$$

$$= 2\sin(t) + 1 - \cos(t) + 4u_3(t)\,\mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{s}{s^2+1}\right\}(t-3)$$

$$\boxed{y(t) = 1 - \cos(t) + 2\sin(t) + 4u_3(t)\left(1 - \cos(t-3)\right)}$$

SAME

$$\boxed{y(t) = \begin{cases} 1 - \cos(t) + 2\sin(t), & t < 3; \\ 5 - \cos(t) + 2\sin(t) - 4\cos(t-3), & t \ge 3. \end{cases}}$$