1. (10 points) Solve the following initial value problem and determine the interval on which the solution is valid.

$$y' = y^2\,(2x+1) \quad y(3) = -1/10$$ (Separable)

$$\frac{1}{y^2}\frac{dy}{dx} = 2x+1$$

$$\int \frac{1}{y^2}\,dy = \int (2x+1)\,dx$$

$$-\frac{1}{y} = x^2+x+c$$

$$y = -\frac{1}{x^2+x+c}$$

$$-\frac{1}{10} = -\frac{1}{9+3+c} \Rightarrow c = -2$$

So $y = \boxed{-\frac{1}{x^2+x-2}}$

This is defined except when $x^2+x-2=0$

$(x+2)(x-1)$

$x=-2$ or $x=1$

intial value at $x=3$, so the solution is valid for $x \in (1,\infty)$

2. (10 points) Ten grams of salt is disolved in a 10 liter tank full of water. Then water containing salt at a concentration of 10 grams per liter trickles in at a rate of 2 liters per hour. The mixed solution flows out of the tank at a rate of 3 liters per hour.

   Determine the concentration (in grams per liter) of salt in the tank at the time when the tank contains 4 liters.

Volume of water $= 10 - t$ Liters

$$Q' = \underbrace{2 \cdot 10}_{\text{in}} - \underbrace{\frac{3Q}{10-t}}_{\text{out}}$$

(at $t=6$, tank contains 4L)

$$Q' + \frac{3Q}{10-t} = 20$$

$$\mu = \exp \int \frac{3}{10-t}\,dt = \exp\left(-3 \ln(10-t)\right) = (10-t)^{-3}$$

$$(10-t)^{-3} Q = \int 20(10-t)^{-3}\,dt = 10(10-t)^{-2} + C$$

$$Q = 10(10-t) + C(10-t)^3$$

$(Q(0) = 10)$

$$10 = 100 + 10^3 C \Rightarrow C = -0.09$$

$$Q = 10(10-t) - 0.09(10-t)^3$$

$$Q(6) = 10(4) - 0.09(4)^3 = 34.24 \text{ g}$$

$$\text{Concentration} = \frac{34.24}{6} = \boxed{5.71 \text{ g/L}}$$

3. (10 points) One solution to the differential equation

$$6t^2y'' + 6ty' - 6y = 0$$

is $y_1 = t$. Use the reduction of order method to find the general solution to this linear homogeneous differential equation.

$y = vt \qquad y' = v't + v \qquad y'' = v''t + 2v'$

$6t^2(v''t + 2v') + 6t(v't + v) - 6vt = 0$

$6t^3v'' + 18t^2v' = 0$

$u = v'$

$6t^3u' + 18t^2u = 0$

$tu' + 3u = 0$

$\frac{u'}{u} = -\frac{3}{t}$

$\ln|u| = -3\ln|t| + c$

$|u| = ce^{-3\ln|t|}$

$u = ct^{-3}$

$v = \int u = \int ct^{-3}dt = c_1t^{-2} + c_2$

$y = vt = \boxed{c_1t^{-1} + c_2t}$

4. (10 points) A 0.5 kg mass stretches a spring by 25 centimeters. A damper with coefficient 6 N/(m/s) is also attached. The spring is pulled down another 25 centimeters and released. Determine the amount of time that elapses before the spring crosses the equilibrium for the first time. Use $g = 9.8$ m/s^2 for acceleration due to gravity.

$m = 0.5 \quad \gamma = 6$

$$mg = KL$$
$$(0.5)(9.8) = 0.25k$$
$$19.6 = k$$

$$0.5u'' + 6u' + 19.6u = 0$$
$$\tfrac{1}{2}r^2 + 6r + 19.6 = 0$$
$$r = \frac{-6 \pm \sqrt{36 - 4(19.6)(\frac{1}{2})}}{2(\frac{1}{2})} = -6 \pm \sqrt{3.2}\, i$$

general: $u = c_1 e^{-6t} \cos\sqrt{3.2}t + c_2 e^{-6t} \sin\sqrt{3.2}t$

$$0.25 = c_1$$
$$u' = -6c_1 e^{-6t}\cos - c_1\sqrt{3.6}e^{-6t}\sin - 6c_2 e^{-6t}\sin + c_2\sqrt{3.2}e^{-6t}\cos$$

$$0 = -6c_1 + \sqrt{3.2}c_2$$
$$c_2 = \frac{1.5}{\sqrt{3.2}}$$

$= 0?$

$$u = e^{-6t}\left[\tfrac{1}{4}\cos\sqrt{3.2}t + \frac{1.5}{\sqrt{3.2}}\sin\sqrt{3.2}t\right]$$

$$\tan\sqrt{3.2}t = -\frac{\sqrt{3.2}}{6} \Rightarrow \sqrt{3.2}t = -0.29 + n\pi$$

(use $n=1$) $\Rightarrow t = \dfrac{\pi - 0.29}{\sqrt{3.2}} = \boxed{1.59}$