Math 300 Exploratory Problem 1: *Where Sums and Products Meet*

Typically, after proving a basic problem, mathematicians ask the question, "Can we generalize this result?". In other words, can we find a more general rule or pattern. As we generalize, we ask more fundamental questions and often encounter more difficult problems as we learn more about the underlying structure of a problem. Write up your answer to the questions that are in **bold** and include this at the end of your homework. This should be a 'fun' additional exercise, so let me know if you are having difficulty.

1. There is only one pair of positive integers x and y that give the same result when added together and when multiplied together (That is, $x + y = xy$).
   Give the pair of numbers (fill in the question marks).
   We can prove that the solution you just gave is the only one as follows.

   > *Proof:* Without loss of generality, assume $x \leq y$. Adding y to both sides gives $x+y \leq 2y$. Since y is positive, if $2 < x$, then multiplying gives $2y < xy$. Thus, if $2 < x$, by the transitive property, $x+y \leq 2y < xy$. Hence, the equation $x+y = xy$ cannot be satisfied when $2 < x$.
   > Thus, $x + y = xy$ can only possibly have solutions when $x \leq 2$, which only leaves $x = 1$ or $x = 2$. Since $x + y = xy$ with $x = 1$ gives $1 + y = y$, there is no solution with $x = 1$. Therefore, only $x = 2$ remains. By solving $2+y = 2y$, we find $y = 2$, so the only solution is $(x, y) =$???. $\square$

   Essential what this shows is that for any large pairs of numbers ($x > 2$) that the product will be larger than the sum, so the only possible solutions are small numbers.

2. Let's jump this up a level and see what we can do.
   There is only one triple of positive integers x, y and z that give the same result when added together and when multiplied together (That is, $x + y + z = xyz$).
   We can follow the reasoning of the proof for the two variable case, with some adjustments, to help us find these numbers.

   > *Proof:* Without loss of generality, assume $x \leq y \leq z$. Thus, $x + y + z \leq z + z + z = 3z$. Since z is positive, if $3 < xy$, then multiplying gives$3z < xyz$. Thus, if $3 < xy$, by the transitive property, $x + y + z \leq 3z < xyz$. Hence, the equation $x + y + z = xyz$ cannot be satisfied when $3 < xy$.
   > Thus, $x + y + z = xyz$ can only possibly have solutions when $xy \leq 3$. Since x and y are positive integers with $x \leq y$, this drastically limits the possibilities to $(x, y) = (1, 1), (1, 2), (1, 3)$. *You will complete the proof by seeing which of these cases leads to a solution with integer z.* $\square$

   For each possibility, solve for the corresponding z, and say which one leads to a three *integer* solution. (*i.e.* give the set of 3 positive integers whose sum and product are equal.)

3. There is also only one solution for the case of four variables ($x + y + z + w = xyzw$). Mimic the proof above for the four variable case in order to prove:
   If (x, y, z, w) is a quadruple of positive integers with $x \leq y \leq z \leq w$ that satisfy $x + y + z + w = xyzw$, then $xyz \leq 4$.
   Then try the possibilities until you **find the set of 4 positive integers whose sum and product are equal.**

Going further (this is for your information, nothing is being assigned below): The five variable case has more than 1 solution, it has three. The six variable case has only 1 solution. The 7, 8, 9, …, 22, and 23 variable cases all have more than 1 solution. But the 24 variable case only has 1 solution. It turns out (by continuing in the way we have done and using a computer to try cases) the only known cases that have unique solutions are the 2, 3, 4, 6, 24, 114, 174, and 444 variable cases. No other cases of unique examples have been found (computers have tested out past the 13 billion variable case). So here is an *unsolved problem*: Prove the 444 variable case is the last case with a unique solution. That is, show that there is only one set of 444 positive integers whose sum and product are the same and for sizes bigger than 444 it is always possible to find more than one set of positive integers whose sum and product are equal.