Math 300 Summer 2009 Final Exam

Name: ______________________________

1	23	
2	26	
3	25	
4	26	
Total	100	

- USE YOUR TIME WISELY! START EACH PROBLEM BY OUTLINING THE PROOFS/IDEAS ON THE PAGE. THE LAST TWO PAGES ARE PROOFS. SPEND NO MORE THAN 10-20 MINUTES ON THE FIRST TWO PAGES.
- Put your answers on the exam, there are 4 pages (you may use your own paper for scratch work). If you need more space for your answer use the back of the preceding page and indicate that you have done so.
- You can pick up your exams anytime during fall quarter (check my website for office hours). Or you can email me in the fall and I will leave your exam in an envelope outside my office.
- If you don't know how to prove something, then you should still let me know everything you know about the problem (that is, give definitions and facts you know about the problem and discuss different techniques to prove the theorem). Also, if you are stuck you should try some examples. Don't leave any question blank, give me the opportunity to give you credit for what you know!

GOOD LUCK!

1. (a) (12 pts) Let $f : A \to B$ be a function. For this function, answer the following questions:

   i. Give the precise definition of one-to-one (with appropriate quantifiers) and give the negation of the definition.
   DEFINITION:

   NEGATION:

   ii. Give the precise definition of onto (with appropriate quantifiers) and give the negation of the definition.
   DEFINITION:

   NEGATION:

(b) (5 pts) Find the truth values of the statement $P \Rightarrow (Q \vee \neg Q)$ for all possible truth values of the statements P and Q (That is, fill in the table below).

P	Q	$\neg Q$	$Q \vee \neg Q$	$P \Rightarrow (Q \vee \neg Q)$
T	T			
T	F			
F	T			
F	F			

(c) (6 pts) TWO of the steps in the 'proof' below are wrong. Circle the two steps that are wrong and briefly explain why they are wrong.
(don't tell me why the theorem is wrong, tell me why the proof is wrong):

False Statement: If $n \nmid y - 1$, then $n \nmid xy - x$ and $n | x$.

False Proof: We prove the contrapositive.
(1) That is, we must show $n | xy - x$ and $n \nmid x$ implies $n | y - 1$.
(2) By definition, $n | xy - x$ is the same as $xy - x \equiv 0 \pmod{n}$.
(3) Adding x to both sides of the congruence gives $xy \equiv x \pmod{n}$.
(4) Since $n \nmid x$, canceling the x gives $y \equiv 1 \pmod{n}$.
(5) Thus, $n | y - 1$. $\square$

EXPLANATION:

2. (a) (12 pts) Consider the statement: If $6b \equiv 0 \pmod{a}$, then $b \equiv 0 \pmod{a}$.

   i. Give a counterexample to this statement (with specific numbers a and b) and add a hypothesis about a that would make the statement true.
      COUNTEREXAMPLE:

      ADDED HYPOTHESIS:

   ii. Prove that the converse of the original statement is always true.

   (b) (14 pts) Prove that $\sum_{k=1}^{n} k \cdot k! = (n+1)! - 1$ for all $n \in \mathbb{N}$.

3. (a) (8 pts) Find an integer a such that $0 \leq a < 5$ and $3^{800} + 4^{801} + 5 \cdot 6^{100} + 7 \equiv a \pmod{5}$. (You must show your work to get credit)

(b) (5 pts) A certain fast food restaurant sells chicken nuggets in packs of either 6 or 9. If these are the only size packs you can buy, is it possible to purchase exactly 94 nuggets? (Specifically, tell me which theorem from class gives us the answer to this question and explain the relationship between these three numbers that allows you to make your conclusion).

(c) (12 pts) Prove that 3 divides $n^3 + 7n$ for all $n \in \mathbb{N}$.

4. (a) (16 pts)

   i. Let p be an odd prime.
   Prove that if $p = a^2 + 1$ for some integer a, then $p \equiv 1 \pmod 4$.
   (Hint: Break your proof into cases, based on a).

   ii. Briefly explain why we can use the theorem above to conclude that all primes of the form $p \equiv 3 \pmod 4$ cannot be written as one more than a square.
   (for example, $11 \equiv 3 \pmod 4$ and 11 is not one more than a square).

   (b) (10 pts) Let $f : A \to B$ and $g : B \to C$ be functions and define $h : A \to C$ by $h(x) = g(f(x))$ for all $x \in A$. Prove if h is one-to-one and f is onto, then g is one-to-one.
   (Hint: You will use the fact that f is a function, so it is well-defined.)