1. (a) For real numbers x and y, consider the statement below and answer the questions.
ORIGINAL: "If $x + y \geq 0$, then $x \geq 0$ or $y \geq 0$."

i. State the contrapositive to the original statement and determine if the original statement is true or false.

(4 pts) CONTRAPOSITIVE:

IF $x < 0$ AND $y < 0$, THEN $x + y < 0$.

(2 pts) CIRCLE THE TRUTH VALUE OF THE ORIGINAL: **TRUE** FALSE

The sum of negative numbers is negative.

ii. State the converse of the original statement and give a counterexample to the converse.

(4 pts) CONVERSE:

IF $x \geq 0$ OR $y \geq 0$, THEN $x + y \geq 0$.

(2 pts) COUNTEREXAMPLE TO CONVERSE:

$x = 3$, $y = -5$ $\rightarrow$ $x \geq 0$ or $y \geq 0$ ✓
$x + y = -2 < 0$

(b) (12 pts) Let A, B, C, and D be sets.
By giving a formal subset proof, show $(A \cap C) \cup (B \cup C)^c \subset A \cup B^c \cup D$.

Assume $x \in (A \cap C) \cup (B \cup C)^c$.

Thus, $x \in A \cap C$ or $x \in (B \cup C)^c$.

CASE 1: If $x \in A \cap C$, then since $A \cap C \subseteq A$ (i.e. in particular $x \in A$) and $A \subseteq A \cup B^c \cup D$, we have $x \in A \cup B^c \cup D$.

CASE 2: If $x \in (B \cup C)^c$, then by deMorgan's law $x \in B^c \cap C^c$. Since $B^c \cap C^c \subseteq B^c \subseteq A \cup B^c \cup D$, we have $x \in A \cup B^c \cup D$.

IN ALL CASES, $x \in A \cup B^c \cup D$. //

2. (a) Give the negation of the following statement (distribute "not" as far as possible). Then determine if it is true (no proof required):
ORIGINAL: For all $a, b \in \mathbb{Z}$, for all $k \in \mathbb{N}$, if ab^k is even, then a is even or b is even.

(6 pts) NEGATION:

THERE EXISTS $a, b \in \mathbb{Z}$ AND THERE EXISTS $k \in \mathbb{N}$ SUCH THAT ab^k IS EVEN AND a IS ODD AND b IS ODD.

(2 pts) CIRCLE WHICH ONE IS **TRUE**: $\boxed{\text{ORIGINAL}}$ NEGATION

If $ab^k = a\underbrace{b \cdots b}_{k \text{ times}}$ is even, the either a or b must be even.

(b) (12 points) Consider the four "proofs" below of the theorem:
Theorem: For all $x, y \in \mathbb{Z}$, if x is odd, then $x + 2y$ is odd.

(Pf 1) $\boxed{\text{Assume } x + 2y \text{ is even}}$ Thus, $x + 2y = 2k$ for some $k \in \mathbb{Z}$. Solving for x gives $x = 2(k - y)$. Hence, $\boxed{x \text{ is even}}$ □	(Pf 2) $\boxed{\text{Assume } x \text{ is odd and } x + 2y \text{ is even}}$ Thus, $x = 2m + 1$ and $x + 2y = 2n$ for some $m, n \in \mathbb{Z}$. Substitution gives $(2m + 1) + 2y = x + 2y = 2n$. Thus, $2k+1+2y = 2n$ which gives $n - k - y = \frac{1}{2}$. $\boxed{\text{Which cannot be true because the sum of integers can't be a fraction.}}$ Hence the original theorem is true. □
PROOF OF CONVERSE (Pf 3) $\boxed{\text{Assume } x + 2y \text{ is odd.}}$ Thus, $x+2y = 2k+1$ for some $k \in \mathbb{Z}$. Solving for x gives $x = 2k - 2y + 1 = 2(k - y) + 1$. $\boxed{\text{Hence, } x \text{ is odd}}$ □	(Pf 4) $\boxed{\text{Assume } x \text{ is odd.}}$ Then $x = 2m + 1$ for some $m \in \mathbb{Z}$. Substitution gives $x + 2y = 2m + 1 + 2y = 2(m + y) + 1$. $\boxed{\text{Thus, } x + 2y \text{ is odd}}$ □

All intermediate steps in the proofs above are correct. At least one of the "proofs" above is an incorrect proof of the stated theorem.
By filling in the table below, for each proof, tell me if it is a correct proof of the given theorem and, if so, tell me which of the main three proof methods is being used.

	CORRECT PROOF OF GIVEN THEOREM? (YES OR NO)	METHOD OF PROOF (IF CORRECT)
(Pf 1)	YES	CONTRAPOSITIVE
(Pf 2)	YES	CONTRADICTION
(Pf 3)	NO	———
(Pf 4)	YES	DIRECT

CHECK YOUR TIME!
LAYOUT THE FORMAT OF YOUR PROOFS FIRST, THEN FILL IN AS MANY DETAILS AS YOU CAN. MAKE SURE YOU LEAVE 15+ MINUTES TO WORK ON THE LAST PAGE.

3. (15 pts) Clearly giving a well structure proof and using the precise definitions of even and odd, prove for all x, y in $\mathbb{Z}$, $xy + x$ is even **if and only if** x is even or y is odd.
(Hint: An indirect method might be useful somewhere in your proof.)

WE MUST SHOW ① $xy+x$ even $\Rightarrow$ x even or y odd

② x even or y odd $\Rightarrow$ $xy+x$ even.

①' We prove the contrapositive.
Assume x is odd AND y is even
Then $x=2m+1$ and $y=2n$ for some $m,n \in \mathbb{Z}$.
By substitution, $xy+x = x(y+1) = (2m+1)(2n+1) = 4mn+2m+2n+1$
$\Rightarrow xy+x = 2(2mn+m+n)+1$ which is odd.

②' Assume x is even or y is odd. (DIRECT PROOF)
CASE 1 If x is even, then $x=2k$ for some $k \in \mathbb{Z}$.
By substitution, $xy+x = x(y+1) = 2k(y+1)$ which is even.
CASE 2 If y is odd, then $y=2l+1$ for some $l \in \mathbb{Z}$
By substitution, $xy+x = x(y+1) = x(2l+2) = 2x(l+1)$, which is even.

In all cases, $xy+x$ is even. //

4. (a) (13 pts) Using induction, prove that for all $n \in \mathbb{Z}$ with $n \geq 0$, $\sum_{i=0}^{n} \pi^i = \frac{\pi^{n+1} - 1}{\pi - 1}$.

$\boxed{\text{BASE STEP}}$ For $n=0$, $\sum_{i=0}^{0} \pi^i = \pi^0 = 1$ and $\frac{\pi^{n+1} - 1}{\pi - 1} = \frac{\pi - 1}{\pi - 1} = 1$. ✓

$\boxed{\text{IND. STEP}}$ Assume $\sum_{i=0}^{k} \pi^i = \frac{\pi^{k+1} - 1}{\pi - 1}$ for some $k \in \mathbb{Z}$, $k \geq 0$.

Then
$$\sum_{i=0}^{k+1} \pi^i = \pi^{k+1} + \sum_{i=0}^{k} \pi^i \quad \text{(PULLING OUT LAST TERM)}$$
$$= \pi^{k+1} + \frac{\pi^{k+1} - 1}{\pi - 1} \quad \text{(BY \underline{INDUCTIVE HYPOTHESIS})}$$
$$= \frac{\pi^{k+1}(\pi - 1)}{\pi - 1} + \frac{\pi^{k+1} - 1}{\pi - 1} \quad \text{(COMMON DENOMINATOR)}$$
$$= \frac{\pi^{k+2} - \cancel{\pi^{k+1}} + \cancel{\pi^{k+1}} - 1}{\pi - 1}$$
$$= \frac{\pi^{k+2} - 1}{\pi - 1}.$$

Thus, the equality holds for all $n \in \mathbb{Z}$, $n \geq 0$. //

(b) (8 pts) Recall: The AGM(a) states $2xy \leq x^2 + y^2$ and AGM(b) states $xy \leq \left(\frac{x+y}{2}\right)^2$ for all real numbers x and y. Prove for all $x, y \in \mathbb{R}$, if $\boxed{x + y \leq 10}$ then $(1 + x)(1 + y) \leq 36$.
(Hint: At some point use one of the AGM inequalities.)

$\boxed{\text{SHOULD BE } 0 \leq x+y \leq 10}$

Since $x+y \leq 10$, we have
$$(1+x)(1+y) = 1 + x + y + xy \leq 1 + 10 + xy = 11 + xy.$$
By AGM(b), $xy \leq \left(\frac{x+y}{2}\right)^2$

Since $x+y \leq 10$, we have $\frac{x+y}{2} \leq 5$
and multiplying this inequality by itself (which is allowed because $\frac{x+y}{2} \geq 0$) gives $\left(\frac{x+y}{2}\right)^2 \leq 25$.

Hence, $xy \leq \left(\frac{x+y}{2}\right)^2 \leq 25$. And putting it all together,
$$(1+x)(1+y) = 1 + x + y + xy \leq 11 + xy \leq 11 + 25 = 36. //$$